Derived homomorphism of LA

NB: • homomorphism between LG = smooth algebraic homomorphism,
• isomorphism between LG = diffeomorphic homomorphism

Def: Homomorphism between L. algebras is linear map $\varphi: \mathcal{G} \to \mathcal{G}'$ such that ($\mathcal{G}$ & $\mathcal{G}'$ are over $\mathbb{F}$)

1, $\varphi(\alpha X + \beta Y) = \alpha\varphi(X) + \beta\varphi(Y)$ $\quad \forall X, Y \in \mathcal{G}$

2, $\varphi([X, Y]) = [\varphi(X), \varphi(Y)]$ $\quad \forall \alpha, \beta \in \mathbb{F}$

Theorem: Let $\phi: G \to H$ be homomorphism between LG and $g(t) = \exp(tX)$, $X \in \mathcal{G}$ is one-param. subgroup in G. Then

1, $\phi(g(t)) = h(t) \subset H$ is one-param. subgroup of H given by

$$h(t) = \exp(tY) \quad Y = \phi_* X \in T_e H$$

2, $\phi_*: T_e G \to T_e H$ is <u>derived homomorphism</u> between $\mathcal{G}$ & $\mathcal{H}$.

• G, H homomorphic $\Rightarrow$ $\mathcal{G}, \mathcal{H}$ homomorphic

• corresponding commutative diagram

$$\begin{tikzcd} G \arrow[r, "\phi"] & H \\ \mathcal{G} \arrow[u, "\exp"] \arrow[r, "\phi_*"] & \mathcal{H} \arrow[u, "\exp"'] \end{tikzcd} \iff \boxed{\phi \circ \exp = \exp \circ \phi_*}$$

Proof:

1, ϕ homomorf $\Rightarrow$ $h(t+s) = \phi(g(t+s)) =$ / g 1-param subgr. /

$= \phi(g(t))\,\phi(g(s)) = h(t)\,h(s)$

$\Rightarrow \exists\, Y \in T_e H :\ h = \exp(tY)$

$$Y(f) \equiv \frac{d}{dt}\left(f(h(t))\right)\Big|_{t=0} = \frac{d}{dt} f(\phi(g(t)))\Big|_{t=0} \equiv (\phi_* X)(f)$$

$\Rightarrow Y = \phi_* X$

2, show that ϕ_* acting on left-invar. field is linear and preserves commutator (exc.)

hint: • $c(\sqrt{t}) = g(\sqrt{t})\,g'(\sqrt{t})\,g(\sqrt{t})^{-1}\,g'(\sqrt{t})^{-1}$ is one-param. subgroup generated by $[X, X'] \in \mathcal{G}$ (Pontryagin)

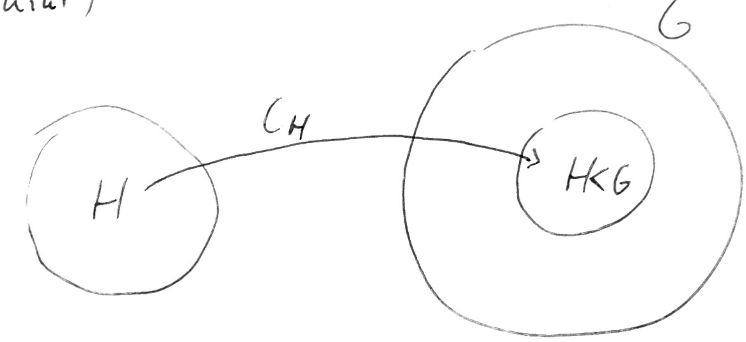
• or use coord. repre

Theorem: Let $H < G$ be a Lie subgroup. Then

$$\mathcal{H} = \{ X \in \mathcal{G} \mid \exp_G(tX) \in H \ \ \forall t \in \mathbb{R} \}$$

is LA of H and subalgebra of $\mathcal{G}$.

Proof: (hint)



$\iota_H : H \hookrightarrow G$ is smooth immersion

$\iota_H(h) = h \in G$

$\Rightarrow (\iota_H)_*$ is derived homomorphism $\mathcal{H} \to \mathcal{G}$

$\Rightarrow \mathcal{H}$ & $\mathcal{G}$ has "the same" commutator

RELATIONS BETWEEN L. groups & L. algebras

- what are L. algebras of isomorphic groups?
- what are L. groups with isomorphic algebras?

<u>Theorem:</u> Let $\phi: G \to G$ be isomorphism between LG. Then the derived homomorphism $\phi_*: \mathcal{G} \to \mathcal{G}'$ is isomorphism.

<u>Proof:</u> 1, ϕ_* is injective:

$\phi_*(X) = \phi_*(Y)$ for $X, Y \in \mathcal{G}$:

$\to \exp(t\phi_*(X)) \equiv \phi(\exp(tX)) = \phi(\exp(tY)) \equiv \exp(t\phi_*(Y))$

$\Rightarrow$ /ϕ injective/ $\to \exp(tX) = \exp(tY) \quad \forall t$

$\to X = Y$ (there is 1-to-1 corresp. between X & $\gamma^X(t)$)

2, ϕ_* is surjective as every $\gamma'(t) \subset G'$ has a pre-image in G (ϕ surjective & homomorphic)

$\Rightarrow$ every $X' \in \mathcal{G}'$ has pre-image in $\mathcal{G}$. $\boxtimes$

<u>Def:</u> <u>Subgroup</u> $H < G$ of a Lie group G is <u>discrete</u> if it is finite or countable & $\exists\, \mathcal{U}(e) \subset G$ which does not contain any element of H other than e.

<u>Theorem:</u> Let $\phi: G \to G'$ ~~be~~ smooth surjective homomorphism between LG & let $\operatorname{Ker}\phi < G$ is a discrete subgroup. Then $\phi_*: \mathcal{G} \to \mathcal{G}'$ is isomorphism.

<u>Proof:</u> (hint)

- $\operatorname{Ker}\phi$ discrete $\to \exists\, \mathcal{U}(e_G): (\operatorname{Ker}\phi) \cap \mathcal{U}(e_G) = \{e_G\}$

$\Rightarrow \phi: \mathcal{U}(e_G) \to \mathcal{U}(e_{G'})$ is isomorphism ($\to \dim G = \dim G'$)

$\Rightarrow \mathcal{G} \sim \mathcal{G}'$

Universal covering group

NB: • center $Z(G) = \{h \in G \mid hg = gh \ \forall g\}$ is abelian normal subgroup of G

• $H < G$ is central if $H \subset Z(G)$

Theorem: Let G be a connected LG. Then there exists a simply connected LG $\bar{G}$ (unique up to isomorphism) such that:

a, G is isomorphic to a (Lie) factor group $\bar{G}/K$, where K is some discrete central subgroup of $\bar{G}$

b, if G is simply connected then $G \sim \bar{G}$

c, $\mathcal{G} \sim \bar{\mathcal{G}}$ (real LA of $G, \bar{G}$)

• $\Rightarrow$ for every LA $\mathcal{G}$ $\exists!$ simply connected "universal covering group"

• K is kernel of some homomorphism, in particular
$$\pi : \bar{G} \to \bar{G}/K \qquad \bar{g} \mapsto \bar{g}K$$

Note: Covering space of a top. space (X, τ) is the topol. space (C, ρ) together with continuous surjective map
$$p : (C, \rho) \xrightarrow{\text{onto}} (X, \tau) \quad \text{such that}$$
$$\forall x \in X \quad \exists\, \mathcal{U}(x) \in \tau : p^{-1}(\mathcal{U}(x)) = \bigcup_\alpha V_\alpha \quad \text{with}$$
$V_\alpha \subset (C, \rho)$, $V_\alpha \cap V_{\alpha'} = \emptyset$ & V_α are homeomorphic to $\mathcal{U}(x)$ through the map p.

$$\begin{tikzcd}
 & & & \boxed{\bar{G}} \arrow[llld, "\pi_1"'] \arrow[lld, "\pi_2"] \arrow[rrd, "\pi_{max}"] \arrow[dd, shift left, "lin"] & & \\
\boxed{G/K_1} \arrow[rrrd, "lin"'] & \boxed{G/K_2} \arrow[rrd, "lin"] & \cdots & & \cdots & \boxed{G/K_{max}} \arrow[lld, "lin"] \\
 & & & \boxed{\mathfrak{g}} \arrow[uu, shift left, "exp"] & &
\end{tikzcd}$$

- K_{max}: max. discrete central subgroup of $\bar{G}$
- finding K_{max} for lin. Lie groups is simple:
  lin. Lie group has faithful finite-dim repre;
  if it is irreducible then $A \in K_{max} \Rightarrow A = \lambda \mathbb{1}$ (Schur)

<u>Example:</u> 1, $SU(2)$ simply connected $\Rightarrow$ it is the universal cover of $su(2) \sim SO(3)$

$$\Rightarrow \overline{SO(3)} = SU(2)$$

- $SO(3) \sim SU(2)/\{-\mathbb{1}, \mathbb{1}\}$ (exercise)

& $\{\mathbb{1}, -\mathbb{1}\}$ is the only nontrivial discrete central subgroup of $SU(2)$

$\Rightarrow$ $SU(2)$ & $SO(3)$ are the only two L. groups with the $su(2)$ algebra; $SU(2) \to SO(3)$ is double covering

2, $\overline{SO(2)} = (\mathbb{R}, +)$

$\pi : (\mathbb{R}, +) \to SO(2)$ $\quad \varphi \mapsto e^{i\varphi}$ is infinite-fold cover $\left(e^{i\varphi} = e^{i(\varphi + 2\pi k)}\right)$

Some structure theory of LA

NB: • subalgebra = vec. subspace closed with resp. to $[\cdot,\cdot]$

• $\mathcal{H} \subsetneq \mathcal{G} \Rightarrow \dim \mathcal{H} < \dim \mathcal{G}$ (at least one generator must be missing

Def: Subalgebra $\mathcal{H} \subset \mathcal{G}$ is invariant (ideal) if
$[X,Y] \in \mathcal{H} \quad \forall X \in \mathcal{H} \quad \forall Y \in \mathcal{G}$

Def: $\mathcal{G}$ is semisimple if it does not contain proper abelian invariant subalgebra.

NB: semisimple LG does not contain proper abelian normal subgroup ($gHg^{-1} = H$) $\rightarrow$ semisimple LA

Def: $\mathcal{G}$ is simple if it is non-abelian and does not contain any proper invar. subalgebra

$\Updownarrow$

NB: simple LG does not contain any proper normal subgroup

(• simple & semi-simple LG important in particle physics
• interesting theoretically – $\exists$ complete classification (cf. Cartan subalgebras, Dynkin diagrams))

Theorem: For a finite-dim. LG G, $H \triangleleft G \Rightarrow \mathcal{H} \subset \mathcal{G}$ is invariant subalgebra.

Proof: • we already know that $\mathcal{H} \subset \mathcal{G}$ is subalgebra (cf. derived homomorphism)

• $H \triangleleft G \Rightarrow C(\sqrt{t}) = h(\sqrt{t})\, g(\sqrt{t})\, h(\sqrt{t})^{-1} g(\sqrt{t})^{-1} \subset H$ 1-param. subgroup & $C(\sqrt{t}) = \exp([A,B]t)$ for $h(t) = e^{tA}$, $g(t) = e^{tB}$
$\Rightarrow [A,B] \in \mathcal{H}$, $A \in \mathcal{H}$, $B \in \mathcal{G}$ arb.

Intermezzo: adjoint representation of LA

Theorem: Let G be real (or complex) LA, $\dim G = n$, & let $e_1, \dots, e_n$ be a basis of G. Define for $X \in G$ $n \times n$ matrix $ad(X)$ by

$$[X, e_j] = \sum_{k=1}^{n} (ad(X))^k_j \, e_k \qquad j = 1, \dots, n$$

Then the matrices $ad(X)$ form n-dimensional adjoint representation of G.

NB: • repre of G ≡ homomorphism $G \to$ matrix algebra preserving $[\cdot,\cdot]$

• $(ad(e_i))^k_j = c^k_{ij}$ ⇒ we have already seen ...

Proof: • $ad(X)$ is well defined: $X \in G \Rightarrow [X, e_j] \in G \Rightarrow [X, e_j] = \sum q^k e_k$

• $[\cdot,\cdot]$ linear ⇒ $ad(\alpha X + \beta Y) = \alpha\, ad(X) + \beta\, ad(Y)$

• $ad([X,Y]) = [ad(X), ad(Y)]$ follows from Jacobi (see (6))

$$[[X,Y], e_j] = (ad([X,Y]))^k_j \, e_k$$

|| Jacobi

$$-[[Y, e_j], X] + [[X, e_j], Y] = -(ad(Y))^k_j [e_k, X] + (ad(X))^k_j [e_k, Y]$$

$$= (ad(Y))^k_j (ad(X))^\ell_k \, e_\ell - (ad(X))^k_j (ad(Y))^\ell_k \, e_\ell$$

$$= [ad(X), ad(Y)]^\ell_j \, e_\ell \qquad \boxtimes$$

- $ad(X)$ <u>is an action of $\mathcal{G}$ on itself</u>:

$$ad: \mathcal{G}\times\mathcal{G}\to\mathcal{G} \qquad ad(X)Y = [X,Y]$$

$$Y = q^j e_j \Rightarrow \cancel{ad}[X,Y] = q^j [X,e_j] = q^j (ad(X))^k_j e_k$$

$$= \tilde{q}^k e_k \Rightarrow \tilde{q} = ad(X)q$$

- $ad(X)$ is lin. operator (matrix) $\Rightarrow$ under a basis transformation $S: e_i \mapsto e_i'$ it transforms as

$$S: ad(X) \mapsto S^{-1} ad(X) S$$

<u>Def</u>: <u>Killing-Cartan form</u> is symmetric bilinear map $B: \mathcal{G}\times\mathcal{G}\to\mathbb{R}/\mathbb{C}$

$$B(X,Y) \equiv Tr[ad(X)ad(Y)] \qquad \forall X,Y\in\mathcal{G}$$

- for a real $\mathcal{G}$, matrix elements $ad(X)\in\mathbb{R}$
  $\to B(X,Y)\in\mathbb{R}$
  ! need not to be true for arb. repre - cf. Pauli matrices
- $\mathcal{G}$ complex $\Rightarrow B(X,Y)\in\mathbb{C}$
- $B(X,Y)$ invariant under all transf. $S\in Aut(\mathcal{G})$
  $\Rightarrow$ is indep. of the basis choice
  $$Tr(S^{-1}ad(X)SS^{-1}ad(Y)S) = Tr(ad(X)ad(Y))$$

<u>Def</u>: <u>Killing-Cartan metric</u> on $\mathcal{G}$ is defined as

$$g_{ij} \equiv B(e_i,e_j) = C^k_{il}\, C^l_{jk} = g_{ji}$$

- transforms as a 2nd-rank tensor
- well defined due to basis independence of $B(\cdot,\cdot)$

Examples: 1, $su(2)$

- $c_{ij}^{k} = \varepsilon_{ijk} \Rightarrow ad(e_i)_j^{k} = \varepsilon_{ijk}$ & $\varepsilon_{ijs}\varepsilon_{kls} = \delta_{ik}\delta_{lj} - \delta_{il}\delta_{jk}$

$\Rightarrow Tr[ad(e_i)\, ad(e_j)] = ad(e_i)_k^{l}\, ad(e_j)_l^{k}$

$= \varepsilon_{ikl}\varepsilon_{jlk} = -\varepsilon_{ikl}\varepsilon_{jkl} = -\delta_{ij}\delta_{kk} + \delta_{ik}\delta_{jk} = -2\delta_{ij}$

$\Rightarrow B(e_i, e_j) = \boxed{g_{ij} = -2\delta_{ij}}$

- g_{ij} is non-degenerate: $\det g = -8 \Rightarrow \nexists X : g(X,Y) = 0\ \forall Y$

$\Rightarrow g_{ij}$ defines inner product on $su(2)$

2, $gl(n,\mathbb{R})$ (exerc.)

- Weyl basis $\rightarrow B(X,Y) = 2n\,Tr(XY) - 2Tr(X)\,Tr(Y)$
- g_{ij} degenerate: $B(\mathbb{1}, Y) = 0\ \forall Y$

3, $sl(n,\mathbb{R})$: $Tr(X) = 0 \Rightarrow B(X,Y) = 2n\,Tr(XY)$

Theorem: (Properties of K-C form on G)

1, $B(Y,X) = B(X,Y)$

2, $B(\alpha X, \beta Y) = \alpha\beta B(X,Y)$

3, $B(X, Y+Z) = B(X,Y) + B(X,Z)$

4, $\psi \in Aut(G) \Rightarrow B(\psi(X), \psi(Y)) = B(X,Y)$

5, $B([X,Y], Z) = B(X, [Y,Z])$

6, G' ideal of G & $B_{G'}$ is K-C form on G'

$\Rightarrow B_{G}(X,Y) = B_{G'}(X,Y) \quad \forall X,Y \in G'$

Theorem (Cartan 1st criterion)

LA $\mathcal{G}$ is solvable $\Leftrightarrow B(X,Y) = 0 \quad \forall X,Y \in \mathcal{G}^{(1)} = [\mathcal{G},\mathcal{G}]$

NB: 1, $\mathcal{G}$ solvable if $\exists n > 0 : D^n \mathcal{G} = 0$, where $D^k \mathcal{G}$ is defined through

- $D^0 \mathcal{G} = \mathcal{G}$ & $D^{k+1}\mathcal{G} = [D^k\mathcal{G}, D^k\mathcal{G}]$
- $[\mathcal{G},\mathcal{G}'] = \{[A,B] / A \in \mathcal{G} \ \& \ B \in \mathcal{G}'\}$

2, $\mathcal{G}$ semi-simple $\Leftrightarrow$ does not contain non-trivial solvable ideal

Theorem (Cartan 2nd criterion)

LA $\mathcal{G}$ semi-simple $\Leftrightarrow B(X,Y)$ non-degenerate

($\Leftrightarrow \det g_{ij} \neq 0$)

Theorem: LG G is compact $\Leftrightarrow B(X,Y)$ on corresp. LA $\mathcal{G}$ is negative definite.

- cf. $su(2) \simeq so(3)$ & $g_{ij} = -2\delta_{ij}$

Corollary: Every compact LG is semi-simple.

$\rightarrow SL(2,\mathbb{R})$ - Gilmore

$\rightarrow$ 54 ...

$\rightarrow$ repre of LA/LG