GROUP ACTION (on a set)

Def: Let G be a group and M a set. It is said that G is acting on M if there exists a mapping

$$\varphi : G \times M \to M \qquad \varphi(g,m) \equiv T(g)m \equiv gm$$

such that $\forall g,h \in G \quad \forall m \in M$

a, $T(g)T(h)m = T(gh)m$

b, $T(e)m = m$

- $T(g)$ is a transformation on M assigned to g
- action is a homomorphism from G to the group of transformations on M (needs not to be injective)

Def: An orbit of an element $m \in M$ under the action of G is the set

$$G.m = \{T(g)m \,/\, g \in G\} \subset M$$

- orbits define equivalence relation on M and partition M to equiv. classes $G.m$

Def: Stabilizer (isotropy) group with respect to $m \in M$ is the subset of G

$$G_m = \{g \in G \,/\, T(g)m = m\} \subset G$$

Lemma: G_m is a subgroup of G

Proof: • $T(e) = \mathbb{1}$ by def. of action $\Rightarrow e \in G_m$

• $g \in G_m \Rightarrow g^{-1} \in G_m$: $m = T(e)m = T(g^{-1}g)m$
$= T(g^{-1})m$

• $g, g' \in G_m \to T(g)T(g')m = m = T(gg')$

$\boxtimes$

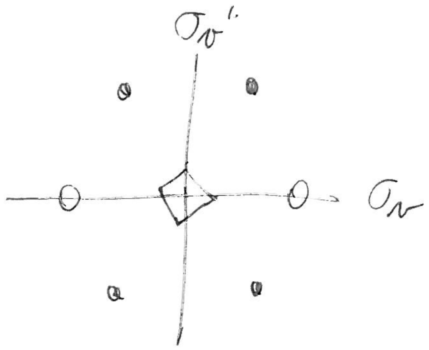
Theorem 10

Let G be a finite group acting on a set M.
Then $(\# G \cdot m) \cdot (\# G_m) = \# G \quad \forall m \in M$.

Proof:
- $n \in G \cdot m \Rightarrow \exists g \in G : T(g) m = n$
- let $m \in M$ and $g \in G$ be fixed and let $\exists g' \in G$:
  $n = T(g) m = T(g') m \Rightarrow T(g^{-1} g') m = m$
  $\Rightarrow g^{-1} g' \in G_m$
  $\Rightarrow$ for fixed g & m there exist exactly $\# G_m$ elements $g' \in G : g^{-1} g' \in G_m$ (rearrang. th.)
- each $g \in G$ maps m to some element from $G \cdot m$
  $\Rightarrow$ it is possible to choose exactly $\#(G \cdot m)$ "non-equivalent" elements of G, each representing the $\# G_m$ elements mapping m to a fixed element of $G \cdot m$
  $\Rightarrow \# G = (\# G \cdot m)(\# G_m)$ $\boxtimes$

- In other words, each element of G maps m somewhere to the orbit and to each element of the orbit maps exactly $\# G_m$ elements of G.

Example: C_{2v} on $\mathbb{R}^3$



• $\Rightarrow \#(G \cdot m) = 4$, $G_m = \{E\}$

∘ $\Rightarrow \#(G \cdot m) = 2$, $G_m = \{E, \sigma_v\}$

◇ $\Rightarrow \#(G \cdot m) = 1$, $G_m = C_{2v}$

Group action on itself: $(G \times G \to G)$

1, left/right translation

$L_g : G \to G \quad h \mapsto gh$

$R_g : G \to G \quad h \mapsto hg$ $\quad \forall h \in G, \; g \in G$ fixed

- $L_g/R_g : G \to G$ is an isomorphism ($\Leftarrow$ rearr. theorem)
- L_g, R_g are transitive : $G.h = G$, $G_h = \{e\}$ for each $h \in G$

2, conjugation (inner automorphism)

$T(g)h \equiv ghg^{-1} \; \forall h \in G$

- $G.h = (h)$

$G_h = \{g \in G / gh = hg\}$ is a subgroup of G

(cf. Proof of theorem 4 - p 8)

REPRESENTATIONS OF GROUPS

- in a simplified way: abstract group → matrix (operator) group ⇒ easier to deal with
- provides many useful informations & tools even without explicit construction of the matrices
- repre need not to be faithful!

recall: • vector space V over the field K
- a set of objects with addition and scalar multiplication by an element from K

• field - set of elements with two binary operations $(+, \cdot)$, both commutative

(• division ring - only + is commutative)

Def: Linear mapping between two vector spaces V and V' is a mapping $A: V \to V'$ satisfying

$$A(\alpha v + \beta w) = \alpha A(v) + \beta A(w) \quad \forall v, w \in V \ \& \ \alpha, \beta \in K$$

- both V & V' must be over the same field K

Def: Linear operator is a linear map $A: V \to V$

- $End(V)$ ... set of all lin. ops. on V (incl. $Ker\, A = 0$)

  ... endomorphisms on V
- $Aut(V)$ ... set of all invertible ($Ker\, A \neq 0$) lin. ops on V

  ... automorphismes on V

  $\sim GL(V)$, linear transformations on V

Def: Representation (ρ, V) of a group G on n-dim vekt. space V over the field K is homomorphism

$$\rho: G \to Aut(V)$$

- $\rho(g_1 g_2) = \rho(g_1)\rho(g_2)$, $\rho(e) = E_{Aut(V)}$, $\rho(g^{-1}) = \rho(g)^{-1}$

- notation
  - ρ is the mapping $G \to \mathrm{Aut}\, V$
  - $\rho(g) \equiv T(g) \in \mathrm{Aut}(V)$ is the lin. op. assigned to $g \in G$
  - $D(g)$ denotes the matrix in the case of matrix representation (i.e., the op. $T(g)$ expressed in a specific basis)

(*)

Def: Let $\{e_1, \dots, e_n\} \subset V$ be a basis of a n-dim. vect. space V. Then each automorphism is given by a matrix $D \in M^{n \times n}$ and, therefore, each $g \in G$ can be associated with a matrix

$$D(g) \in GL(n, K)$$

The mapping $D: G \to GL(n, K)$ is called <u>matrix representation</u> D of a group G

- V needs not be specified anymore
- matrix corresponding to an op. $T(g)$:

D^{i}_{j} ← row (i), ↖ col. (j)

$$x = e_i x^i \quad \dots \quad e_i \text{ - basis vectors}, \quad x^i \text{ - coordinates}$$

- $\boxed{T(g)\, e_i \equiv e_k\, D(g)^k_{\ i}}$
  - note the order of e_k & $D(g)$: e_k is a "row vector"

- $x' = T(g)x = T(g)\, e_i x^i = e_k\, D(g)^k_{\ i}\, x^i = e_k\, x'^k$

$$\Rightarrow \boxed{x'^k = D(g)^k_{\ i}\, x^i}$$

- this is std. matrix-vector multiplication

- exercise: $D(e) = \mathbb{1}$ & $D(g_1 g_2) = D(g_1) D(g_2)$

  $\Rightarrow D$ is a representation

(*)

Def: The vector space V is called the <u>representation space</u> of ρ and the dimension of V is the <u>dimension of the repre.</u>

Def: <u>Trivial representation</u> $\rho(g) = \mathbb{1}_{Aut(V)} \quad \forall g \in G$

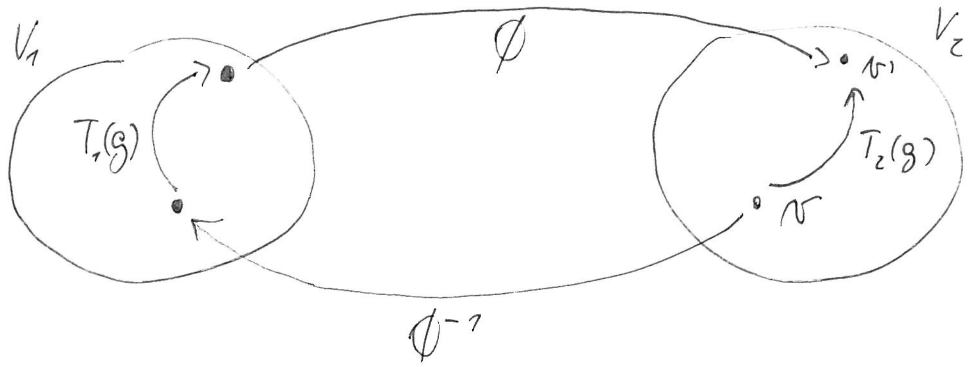
- for $\dim V = 1$, triv. repre is called <u>totally symmetric irreducible representation</u>

Def: If $\rho : G \to Aut(V)$ is injective, the <u>representation</u> (ρ, V) is called <u>faithful</u>

Notes:
- there $\exists$ infinitely many repre of a given G on vect. spaces of various dimensions
- even on the same V there might exist different repre's, many are, however, <u>equivalent</u>
- some of the multi-dimensional representations can be decomposed to several less-dimensional $\Rightarrow$ <u>reducibility</u>

Def: Let V_1 and V_2 be vector spaces. Two <u>representations</u> (ρ_1, V_1) and (ρ_2, V_2) of a group G are called <u>equivalent</u> if there exist an isomorphic mapping $\phi : V_1 \to V_2$ such that

$$T_2(g)v = \phi \cdot T_1(g) \cdot \phi^{-1} v \quad \forall v \in V_2 \text{ and } \forall g \in G.$$



$$\boxed{\rho_1 \sim \rho_2}$$

Def: <u>Intertwining map</u> $S : V_1 \to V_2$ is a map that for two repre (ρ_1, V_1) & (ρ_2, V_2) satisfies

$$S \cdot T_1(g) v = T_2(g) \circ S v \quad \forall v \in V_1 \ \& \ \forall g \in G$$