(25)

Note: • $\rho_1 \sim \rho_2 \Leftrightarrow \exists S$ intertwining isomorphism $(\exists S^{-1})$

• $\dim V_1 = \dim V_2$ is not sufficient condition for equivalence of ρ_1 & ρ_2

Examples • see homomorphisms $C_{2v} \to M^{3\times 3}$ & $C_{2v} \to (\{1,-1\},\cdot)$ discussed before

• $C_i = \{E, i\}$, $\rho: C_i \to Aut(\mathbb{R})$

1, $\rho_g(E) = \rho_g(i) = 1$ $\qquad g \leftrightarrow$ "gerade"

2, $\rho_u(E) = 1$, $\rho_u(i) = -1$ $\qquad$ "ungerade"

Theorem 11: Let two matrix representations D and $\tilde{D}$ of a group G be connected by a similarity transformation,

$$\tilde{D}(g) = B D(g) B^{-1} \quad \forall g \in G.$$

Then $D \sim \tilde{D}$.

Proof: • lets study basis transformation:

• $T(g)\, e_i = e_k D(g)^k_{\ i} \Rightarrow x'^k = D(g)^k_{\ i} x^i$

• basis transformation: $\hat{e}_i = e_j A^j_{\ i} \Leftrightarrow e_j = \hat{e}_i (A^{-1})^i_{\ j}$

$A: V \to V$ ... transformation matrix, $\det A \neq 0$

$$x = \hat{e}_i \hat{x}^i = e_j x^j = \hat{e}_i (A^{-1})^i_{\ j} x^j \Rightarrow \hat{x}^i = (A^{-1})^i_{\ j} x^j \equiv B^i_{\ j} x^j$$

$$\Rightarrow \boxed{\hat{x} = B x}$$

• $e_i \leftrightarrow$ repre $D(g)$: $x' = D(g) x$

$\hat{e}_i \leftrightarrow$ repre $\hat{D}(g)$: $\hat{x}' = \hat{D}(g) \hat{x}$ $\qquad$ & $\hat{x}' = B x'$ for $\hat{x} = Bx$

$$\hat{x}' = \underbrace{\hat{D}(g) \hat{x}} = \hat{D}(g) B x = B x' = B D(g) x = \underbrace{B D(g) B^{-1} \hat{x}}$$

$$\Rightarrow \boxed{\hat{D}(g) = B D(g) B^{-1}}$$

- $A = B^{-1}$ defines isomorphism $V \to V$ $\Rightarrow$ D & $\hat{D}$ are equivalent ☒

REDUCIBLE & IRREDUCIBLE REPRESENTATIONS

Def: Let $\phi: G \times V \to V$ be an action of a group G on a vect. space V and let $W \subset V$ is preserved under the action, that is,

$$T(g)w \in W \quad \forall g \in G \quad \forall w \in W \qquad (G \cdot W \subset W).$$

Then the subspace $W \subset V$ is called invariant.

Def: Inv. subspace W is called irreducible if it does not contain any other non-trivial invariant subspace.

Def: Let V contains an invariant subspace under the group action defined by a repre (ρ, V). Then the vect. space V is called reducible and (ρ, V) is reducible representation. Representation that is not reducible is called irreducible.

Def: A subrepresentation of a representation (ρ, V) is a representation $(\rho_{\upharpoonright W}, W)$, where $W \in V$ invariant subspace under the group action def. by (ρ, V).

Reducible matrix representations

- let $W \subset V$ be an invariant subspace $\Rightarrow$ consider basis

$$W = \mathrm{span}(e_1, \dots, e_r) \qquad \text{span} \dots \text{linear span}$$
$$W_\perp = V \setminus W = \mathrm{span}(e_{r+1}, \dots, e_d) \qquad \setminus \dots \text{complement}$$

- W invariant $\Rightarrow$ $T(g)\, e_i = \sum_{k=1}^{r} e_k D(g)^k{}_i = \sum_k e_k D^W(g)^k{}_i$

$$\Rightarrow D(g) = \left(\begin{array}{c|c} D^W(g) & D^{WW_\perp}(g) \\ \hline 0 & D^{W_\perp}(g) \end{array}\right) \qquad (*)$$

- $D^W(g)$ forms a subrepre of $D(g)$:

$$D(g_1)D(g_2) = \left(\begin{array}{c|c} D^W(g_1)D^W(g_2) & \dots\text{exercise}\dots \\ \hline 0 & D^{W_\perp}(g_1)D^{W_\perp}(g_2) \end{array}\right)$$

- $D^{W_\perp}(g)$ is also repre of G, but it is not (in general) subrepre of G … $W_\perp$ needs not be invariant

NOTE:
- in an arbitrary basis of V, which does not conform to the structure of invariant subspaces, the matrices $D(g)$ do not have the block structure $(*)$
- however, $\exists$ similarity transform which will convert $D(g)$ to this form $\forall g \in G$ $\Downarrow$

Def: Matrix representation is said to be reducible if it is equivalent to a matrix repre $D(g)$, in which the matrices have block structure $(*)$ $\forall g \in G$.

Examples

a, 2-dim repre of $G(\mathbb{R}^+, \cdot)$

- $(\mathbb{R}^+, \cdot) \sim (\mathbb{R}, +)$: $x = e^y$ ... $x \in (\mathbb{R}^+, \cdot)$, $y \in (\mathbb{R}, +)$

$$x x' = e^{y+y'} \; ; \; y = \log(x)$$

$$\Rightarrow D(x) = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \; : \; D(x)D(x') = \begin{pmatrix} 1 & y+y' \\ 0 & 1 \end{pmatrix} = D(x, x')$$

- $D(x)$ is reducible ; $W = \mathrm{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$

$D^W(x) = 1 \; \forall x$ ... triv. representation

b, 3-dim repre of $SO(2)$

- action of $SO(2)$ on $\mathbb{R}^3$: $T(g) v = R^z_\varphi v$

$$\left.\begin{array}{l} G \cdot \mathrm{span}(e_z) = \mathrm{span}(e_z) = W_1 \\ G \cdot \mathrm{span}(\{e_x, e_y\}) = \mathrm{span}(\{e_x, e_y\}) = W_2 \end{array}\right\} \text{both invariant}$$

$$R^z_\varphi = \begin{pmatrix} c\varphi & -s\varphi & 0 \\ s\varphi & c\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\nearrow$ $R^z_\varphi \downarrow W_1 = 1$ ... triv. repre

$\searrow$ $R^z_\varphi \downarrow W_2 = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$ ... faithful 2D repre

- $T(g) = R^x_\varphi \Rightarrow$ different decomposition to inv. subspaces

$(\{e_x\} \cup \{e_y, e_z\})$

Def: Repre (ρ, V) of a group G is called <u>completely reducible</u> if the repre. space V is a direct sum of irreducible invariant subspaces, $V = \bigoplus_i V_i$, under the group action

- <u>direct sum</u> of vect. spaces:

$$V = U \cup W \;\&\; U \cap W = \{0\} \Rightarrow V = U \oplus W$$

$\Rightarrow v = u + w$ is unique decomposition, $u \in U$ & $w \in W$

- in an appropriate basis complying with the decomposition the matrices of the representation have block-diag. form

$$D(g) = \begin{pmatrix} D^1(g) & & 0 \\ & D^2(g) & \\ 0 & & \ddots & D^p(g) \end{pmatrix} \equiv \mathrm{diag}\left(D^1(g), D^2(g), \dots, D^p(g)\right) \equiv \bigoplus_i D^i(g)$$

$\Rightarrow D^i(g)$ are irreducible repre

Def: Matrix repre $D(g)$ is completely reducible if it is equivalent to a repre $D'(g)$ of block-diagonal matrices $\forall g \in G$.

Examples:

- $SO(2)$ on $\mathbb{R}^3$ is completely reducible after generalization to complex space:

$\xi_1 = e_x + ie_y \quad \xi_2 = e_x - ie_y$ generate 1D invariant subspaces:

NOTE (later): complex irred. repre of an abelian gr. are 1D

$$\begin{pmatrix} c\varphi & -s\varphi \\ s\varphi & c\varphi \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = e^{-i\varphi}\begin{pmatrix} 1 \\ i \end{pmatrix}, \dots$$

$$A = \frac{i}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \Rightarrow A^+\begin{pmatrix} c\varphi & -s\varphi \\ s\varphi & c\varphi \end{pmatrix}A = \begin{pmatrix} e^{-i\varphi} & 0 \\ 0 & e^{i\varphi} \end{pmatrix}$$

$\Rightarrow e^{-i\varphi}$ & $e^{i\varphi}$ are two non-equiv. faithful 1D repre

- $D(x) = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$ as a repre of $(\mathbb{R}^+, \cdot) \sim (\mathbb{R}, +)$ is not completely reducible:

Let $\exists A$: $\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = A\begin{pmatrix} f(y) & 0 \\ 0 & g(y) \end{pmatrix}A^{-1}$ $\Rightarrow$ / det & Tr are invariant under similarity transf / $\Rightarrow$ $f(y)g(y) = 1$
$f(y) + g(y) = 2$

$\Rightarrow f(y) = g(y) = 1 \quad \forall y$ (indep. of y)

Theorem XII

Any irreducible representation of a finite group is finite-dimensional.

Proof: • (ρ, V) is irrep, $x \in V$ arbitrary

$G \cdot x = \{T(g)x \,/\, g \in G\}$ is finite-dim set of vectors from V

$\Rightarrow$ span$(G \cdot x) \subset V$ is finite-dim invar. subspace of V

• (ρ, V) irrep $\Rightarrow$ span$(G \cdot x) = V \Rightarrow \dim V < +\infty$ $\boxtimes$

UNITARY REPRESENTATIONS

• Hilbert space:
- vect. space $\mathcal{H}$ with an inner (dot) product $\langle \cdot | \cdot \rangle$
- it is complete with respect to the metric induced by the inner product (Cauchy sequence is convergent within $\mathcal{H}$)
- it is separable (each $\psi \in \mathcal{H}$ is a limit of some sequence from a countable subset $M \subset \mathcal{H}$ $\Leftrightarrow$ $\exists$ countable basis)

• unitary operators:
- for bounded operators $(\exists K \in \mathbb{R} : \|U\psi\| \leq K\|\psi\| \;\; \forall \psi \in \mathcal{H})$ it is possible to define conjugation:

$$\langle \psi | A^+ \varphi \rangle \equiv \langle A\psi | \varphi \rangle \qquad \forall \psi, \varphi \in \mathcal{H}$$

- then for invertible bounded ops we can define unitarity:

$$\langle \psi | \varphi \rangle = \langle U\psi | U\varphi \rangle = \langle \psi | U^+ U \varphi \rangle \qquad \forall \psi, \varphi \in \mathcal{H}$$

$$\Leftrightarrow U^+ U = \mathbb{1} \;\; \& \;\; \exists U^{-1} \Rightarrow U^{-1} = U^+ \Rightarrow UU^+ = \mathbb{1}$$

Def: Unitary repre of G is a repre on a Hilbert space $\mathcal{H}$ such that every $g \in G$ is represented by an unitary operator $U(g)$:

$$U(g)^+ U(g) = U(g) U(g)^+ = \mathbb{1} \Leftrightarrow \langle U(g)\psi | U(g)\varphi \rangle = \langle \psi | \varphi \rangle$$
$$\forall g \in G \;\; \& \;\; \forall \psi, \varphi \in \mathcal{H}$$

Def: Matrix unitary repre is such that every element $g \in G$ is represented by an unitary matrix
$$D(g)^{-1} = D(g)^{+}$$

Theorem XIII

Every finite-dimensional reducible unitary repre $(\rho, \mathcal{H})$ of a group G is completely reducible.

Proof: • ρ reducible $\Rightarrow \exists\, \mathcal{H}_1 \subset \mathcal{H}$ nontriv. inv. subspace

$\Rightarrow \mathcal{H}_1^{\perp} = \mathcal{H} \setminus \mathcal{H}_1$ is also invariant:

• $\psi \in \mathcal{H}_1^{\perp} \Rightarrow \langle \varphi | \psi \rangle = 0 \quad \forall \varphi \in \mathcal{H}_1$

• $U(g)\psi \in \mathcal{H}_1^{\perp}$: $\langle \varphi | U(g)\psi \rangle = \langle U(g)U(g)^{-1}\varphi | U(g)\psi \rangle$

$= \langle U(g^{-1})\varphi | U(g)^{+}U(g) \rangle = \langle \varphi' | \psi \rangle = 0$

$\mathcal{H}_1$ inv. $\Rightarrow U(g^{-1})\varphi \in \mathcal{H}_1$

• if $\mathcal{H}_1$ or $\mathcal{H}_1^{+}$ are further reducible then the same decomposition can be repeated until complete reducibility ... provided $\dim \mathcal{H} < +\infty$ ⊠

Note: 2D-repre $(\mathbb{R}^{+}, \cdot)$ is not unitary $\Rightarrow$ Th. does not apply

Theorem XIV

Every finite-dim representation of a finite or compact Lie group is equivalent to some unitary representation.

NB: we do not require the repre space to be a Hilbert space!

Proof (hint):

- every finite-dim vect. space ~ $\mathbb{R}^n$ or $\mathbb{C}^n$
  $\Rightarrow$ it is possible to select basis $\{e_1, \dots, e_n\}$

  $\Rightarrow$ $x = x^i e_i \quad \forall x \in V$

  $\Rightarrow$ $\langle x | y \rangle \equiv (x^i)^* y^i$ is a legal inner product:

  - $\langle x | y \rangle = \langle y | x \rangle^*$ (conjugate sym.)
  - linear in first argument
  - $\langle x | x \rangle > 0 \quad \forall x \in V \setminus \{0\}$ (positive definite)

- if (ρ, V) is not unitary with resp. to $\langle \cdot | \cdot \rangle$, it is possible to construct another inner product

$$(x|y) \equiv \frac{1}{\#G} \sum_{g'} \langle T(g')x | T(g')y \rangle$$

$\Rightarrow$ /rearr. theorem/ $\Rightarrow$ $(T(g)x | T(g)y) = (x|y)$

$\Rightarrow$ (ρ, V) is unitary with resp. to $(\cdot|\cdot)$ $\boxtimes$

Note: • $(\cdot|\cdot)$ is equivalent to $\langle \cdot | \cdot \rangle$ – induces equal topology (ie., open sets on V) $\Leftrightarrow$ the same convergent sequences

$\Leftrightarrow$ $\exists a, b \in \mathbb{R}, \ 0 < a \leq b: \ a \langle x|x \rangle \leq (x|x) \leq b \langle x|x \rangle \quad \forall x \in V$

- <u>compact Lie groups</u> = compact smooth manifolds
  = parametrized by coordinates from a compact subspace of $\mathbb{R}^n$ if $\exists$ global map;
  (<u>compact set</u> in $\mathbb{R}^n$: closed & bounded)

  – $SO(2)$, $O(n)$ are compact, $(\mathbb{R}^+, \cdot)$ is not

- for comp. Lie groups $\exists$ <u>left-invariant measure</u>

$$\exists \int_G dg < +\infty : \int_G f(hg)\, dg = \int_G f(g)\, dg \Rightarrow \frac{1}{\#G} \sum_g \to \frac{1}{|G|} \int_G dg$$

Theorem XV (Maschke)

Every finite-dim reducible repre of a finite (compact Lie) group is completely reducible.

Proof: • combine Th XIV & XIII $\boxtimes$

SCHUR's LEMMAS

Lemma SL1:

Let (ρ_1, V_1) and (ρ_2, V_2) be irreducibilni repre of a group G & let S is the intertwining mapping

$$S: V_1 \to V_2 \qquad S\, T_1(g)\, v_1 = T_2(g)\, S\, v_1 \quad \forall g \in G \;\; \forall v_1 \in V_1$$

Then either $S = 0$ ($\Leftrightarrow \operatorname{Ker} S = V_1$) or S is isomorphic map & $\rho_1 \simeq \rho_2$.

Proof: • $\operatorname{Ker} S$ & $\operatorname{Im} S$ are invariant subspaces of V_1 & V_2, resp:

a) $v_1 \in \operatorname{Ker} S \Rightarrow S\, T_1(g)\, v_1 = T_2(g)\, S v_1 = 0 \Rightarrow T_1(g) v_1 \in \operatorname{Ker} S$

b) $v_2 \in \operatorname{Im} S \Rightarrow \exists v_1 \in V_1 : v_2 = S v_1$

$\Rightarrow T_2(g) v_2 = T_2(g)\, S v_1 = S\, T_1(g)\, v_1 = S v_1' \Rightarrow T_2(g)\, v_2 \in \operatorname{Im} S$

• V_1 & V_2 are irreducible $\Rightarrow$ there are only 2 options:

a) $\operatorname{Ker} S = V_1$ & $\operatorname{Im} S = \{0\} \Leftrightarrow S = 0$

b) $\operatorname{Im} S = V_2$ & $\operatorname{Ker} S = \{0\} \Rightarrow S$ is bijective:

• surjective by $\operatorname{Im} S = V_2$

• injective: $S v_1 = S v_1' = v_2 \Rightarrow S(v_1 - v_1') = 0$

$\Rightarrow v_1 - v_1' \in \operatorname{Ker} S \Rightarrow v_1 - v_1' = 0$

$\boxtimes$

Lemma SL2 (consequence of SL1 for finite-dim irreps)

Let (ρ, V) be a complex finite-dim irreducible repre of a group G and S is intertwining operator on V ($S: V \to V$) such that

$$T(g) S v = S T(g) v \quad \forall g \in G \ \forall v \in V.$$

Then $S = \lambda \mathbb{1}$ for some $\lambda \in \mathbb{C}$.

Proof: • let $S \neq 0 \Rightarrow \exists \lambda \in \mathbb{C} \ \exists v_\lambda \in V: S v_\lambda = \lambda v_\lambda$
(in a finite dim there always exists a solution of a characteristic polynomial in $\mathbb{C} \Rightarrow$ every operator has an eigenvalue)

• eigen-subspace $V_\lambda \subset V$ corresponding to λ is invariant under the action of G:

$$v \in V_\lambda \Rightarrow S T(g) v = T(g) S v = \lambda T(g) v \Rightarrow T(g) v \in V_\lambda$$

$$\Rightarrow /(\rho, V) \text{ irred.}/ \Rightarrow V_\lambda = V \Rightarrow S = \lambda \mathbb{1} \quad (S e_i = \lambda e_i \ \forall e_i \in \text{basis})$$

Note: • in an infinite-dim V the subspace V_λ need not to be closed:

$\{v_k\} \subset V_\lambda$ Cauchy sequence in V_λ does not imply $\lim_{k\to\infty} v_k = v \in V_\lambda \Rightarrow S T(g) = T(g) S$ does not imply invariance of V_λ

CONSEQUENCES OF SCHUR LEMMAS

Theorem XVI

Complex finite-dim irreducible representations of an Abelian group are one-dimensional.

Proof: • (ρ, V) finite-dim, G Abelian $\Rightarrow$

$$T(g)\,T(h) = T(h)\,T(g) \quad \forall g, h \in G$$

$$\overset{(SL2)}{\Rightarrow} T(h) = \lambda(h)\,\mathbb{1} \quad \forall h \in G$$

$\Rightarrow \rho$ is either reducible or 1-dim. $\boxtimes$

Note: $SO(2)$ is Abelian but we had to move into complex repre space to obtain 1-dim irreps.

Theorem XVII (Orthogonality relations for matrix representations)

Let D^μ and D^ν are two unitary irreducible matrix repre of a finite (compact Lie) group. Dimensions of the representations are d_μ & d_ν, resp. Let D^μ and D^ν are not equivalent for $\mu \neq \nu$ or identical for $\mu = \nu$. Then

$$\sum_g [D^\mu(g)^*]^j_i\, D^\nu(g)^k_l = \frac{\#G}{d_\mu}\,\delta_{\mu\nu}\,\delta_{kj}\,\delta_{il} \qquad (*)$$

Proof: • B is an arbitrary $d_\mu \times d_\nu$ matrix

$$\Rightarrow A = \sum_g D^\mu(g^{-1})\,B\,D^\nu(g) \quad \Rightarrow D^\mu(h)\,A = A\,D^\nu(h) \quad \forall h \in G:$$

$$\sum_g D^\mu(h)\,D^\mu(g^{-1})\,B\,D^\nu(g) = /\, hg^{-1} \equiv g'^{-1} \Rightarrow g' = gh^{-1} \text{ \& rearr. theorem} /$$

$$= \sum_{g'} D^\mu(g'^{-1})\,B\,D^\nu(g'h) = A\,D^\nu(h)$$

1, D^μ not equiv to D^ν $\Rightarrow$ /SL2/ $\Rightarrow$ $A \equiv 0$ & choose

$B^j_k = \delta_{jr}\delta_{ks}$ for a fixed r, s

$$\Rightarrow 0 = \sum_{jk}\sum_g D^\mu(g^{-1})^i_j\,\delta_{jr}\delta_{ks}\,D^\nu(g)^k_\ell = \sum_g D^\mu(g^{-1})^i_r\,D^\nu(g)^s_\ell$$

$$\boxed{0 = \sum_g \left(D^\mu(g)^r_i\right)^* D^\nu(g)^s_\ell \quad \text{for } \mu \neq \nu}$$

2, $D^\mu \sim D^\nu \Rightarrow \exists S: \; D^\nu(g) = S D^\mu(g) S^{-1}$

$\Rightarrow D^\mu(h) A = A S D^\mu(h) S^{-1} \to D^\mu(h) A S = A S D^\mu(h)$

$\Rightarrow$ /SL2/ $\Rightarrow$ $AS = \lambda \mathbb{1}_{d_\mu \times d_\mu}$

- λ from Tr: $Tr(AS) = d_\mu \lambda = Tr\left(\sum_g D^\mu(g^{-1}) B S D^\mu(g) S^{-1} S\right)$

$$\Rightarrow \lambda = \frac{\#G}{d_\mu} Tr(BS) = /B^j_k = \delta_{jr}\delta_{ks}/ = \frac{\#G}{d_\mu} B^j_k S^k_j$$

$$\Rightarrow \boxed{\lambda = \frac{\#G}{d_\mu} S^s_r}$$

- $A^i_\ell = \lambda (S^{-1})^i_\ell = \sum_{jk}\sum_g D^\mu(g^{-1})^i_j\,\delta_{jr}\delta_{ks}\,D^\nu(g)^k_\ell$

$$\Rightarrow \sum_g \left(D^\mu(g)^r_i\right)^* D^\nu(g)^s_\ell = \frac{\#G}{d_\mu} S^s_r (S^{-1})^i_\ell$$

3, $\mu = \nu \Rightarrow S^i_\ell = (S^{-1})^i_\ell = \delta_{i\ell}$

$$\Rightarrow \sum_g \left(D^\mu(g)^r_i\right)^* D^\nu(g)^s_\ell = \frac{\#G}{d_\mu}\delta_{\mu\nu}\delta_{sr}\delta_{i\ell} \quad \boxtimes$$

Direct consequence: $\sum_\mu d_\mu \leq \#G$:

(*) is orthogonality relation between #G-dim vectors $\left(D^\mu(g_1)^i_j, \dots, D^\mu(g_{\#G})^i_j\right)$, which are indexed by (μ, i, j)

$\Rightarrow$ total muber of these orthog. vectors is $\sum_\mu d_\mu^2$ & must be $\leq \#G$