CHARACTER OF A REPRESENTATION

- goal: find a property which will be equal for equiv. repre's and, if possible, different for non-equiv repre's (the second goal will be met only for finite groups)

$\Rightarrow$ for matrix repre, we are looking for invariants with resp. to similarity transforms

a, all eigenvalues

b, Tr

Def: Let (ρ, V) be a repre of a gr. G on a finite-dim V and let D be a corresp. matrix repre in some basis. Then the function $\chi : G \to \mathbb{C}$

$$g \mapsto \mathrm{Tr}\, D(g)$$

is called character of a representation (also character system) and the number $\chi(g) = \mathrm{Tr}\, D(g)$ is character of an element $g \in G$ in a repre (ρ, V).

Notes:

- $\chi(g') = \chi(g) \quad \forall g' \in (g) \quad \Leftarrow \mathrm{Tr}(ABC) = \mathrm{Tr}(CAB)$

  $\Rightarrow$ character of an element is character of the whole class
- χ is equal for all equiv. repre's $\Rightarrow$ it is a characteristics of the equivalence class
- equal χ does not imply equiv. repre:

  $(\mathbb{R}^+, \cdot)$: $D(g) = \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix}$ not equiv. to $\hat{D}(g) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- for finite groups, we will show $\chi = \chi' \Rightarrow D \sim D'$
- $\chi(e) = \dim V \qquad (\Leftarrow D(e) = \mathbb{1})$
- $\chi(g^{-1}) = \chi(g)^*$ for finite-dim repre ($\Leftarrow$ Th. XIV & equivalence to a unitary repre.)

- character of a reducible representation (block-diag. matrices) completely

$$\boxed{D(g) = \bigoplus_i D^i(g) \Rightarrow \chi(g) = \sum_i \chi^i(g)}$$

Theorem XVIII (orthog. relations for χ)

Let χ^μ and χ^ν be characters of two IRREPs of a finite (or compact Lie) group on a complex finite-dim vect. spaces and let the IRREPs are non-equiv for $\mu \neq \nu$.

Then

$$\boxed{\sum_g \chi^\mu(g^{-1})\,\chi^\nu(g) = \sum_g \chi^\mu(g)^* \chi^\nu(g) = \#G\,\delta_{\mu\nu}}$$

- for compact Lie groups: $\sum_g \to \int_G dg$

Proof: directly from Th. XVII:

- $\sum_g D^\mu(g^{-1})^i_j\, D^\nu(g)^k_l = \frac{\#G}{d_\mu}\delta_{\mu\nu}\,\delta_{il}\,\delta_{kj} \Rightarrow / i=j \;\&\; k=l\;,\; \sum_{ik} / \Rightarrow$

$\Rightarrow \sum_g \chi^\mu(g^{-1})\,\chi^\nu(g) = \frac{\#G}{d_\mu}\delta_{\mu\nu}\sum_{ki}(\delta_{ik})^2 = \#G\,\delta_{\mu\nu}$

- $\chi^\mu(g^{-1}) = \chi^\mu(g)^*$: ρ^μ ~ unit. repre $\Rightarrow \lambda_i^{-1} = \lambda_i^*$ for all eigenvalues (eigenvalues of unitary op/matrix are complex units: $|\lambda_i|=1$)

$\boxtimes$

Note:

- $\chi(g') = \chi(g)\;\; \forall g' \in (g) \Rightarrow$ orthog. relations can be written using characters of classes:

$$\boxed{\sum_{k=1}^{N_c} n_k\,\chi^\mu(g_k)^*\,\chi^\nu(g_k) = \#G\,\delta_{\mu\nu}} \qquad (+)$$

- $k = 1, \dots, N_c$ numbers all distinct classes (g_k)
- $n_k = \#(g_k)$

- direct consequence : $\boxed{\# \text{IRREPs} \leq N_c}$
  $\Leftarrow$ (+) says that characters of non-equiv. IRREPs form a set of orthog. vectors in N_c-dim vect. space
- we will prove later that there is in fact "="

Theorem XIX:
Let G be a finite or compact Lie group. Then equality of characters of two representations is sufficient condition for their equivalence.

Proof: 1, let ρ^μ and ρ^ν are two non-equiv IRREPs with equal characters

$$\overset{\text{non}}{\underset{\text{equiv}}{\Rightarrow}} \sum_g \chi^\mu(g)^* \chi^\nu(g) = 0 \quad \& \quad \chi^\mu(g) = \chi^\nu(g) \ \forall g$$

$$\Rightarrow \sum_g \chi^\mu(g)^* \chi^\mu(g) = \#G \qquad \text{↯}$$

2, let ρ^i & ρ^j are reducible $\Rightarrow$ /Maschke/
$\rightarrow$ they are completely reducible

$$\Rightarrow \rho^i = \bigoplus_\alpha n^i_\alpha \rho^\alpha \qquad \rho^j = \bigoplus_\alpha n^j_\alpha \rho^\alpha$$

- here ρ^α are all IRREPs of G, we already know it is a finite expansion)
- $\rho^i = \bigoplus_\alpha n^i_\alpha \rho^\alpha$ means that the repre space V^i contains n^i_α-times repre space V^α as a invar. subspace
  $\rightarrow$ in terms of matrices:

$$D^i = \mathrm{diag}(\ldots, \underbrace{D^\alpha, D^\alpha, \ldots, D^\alpha}_{n^i_\alpha\text{-times}}, \ldots)$$

$$\Rightarrow \chi^i(g) = \sum_\alpha n^i_\alpha \chi^\alpha(g) \qquad \chi^j(g) = \sum_\alpha n^j_\alpha \chi^\alpha(g)$$

- by asumption, $\chi^i(g) = \chi^j(g) \quad \forall g$

$$\to \sum_\alpha (n^i_\alpha - n^j_\alpha)\, \chi^\alpha(g) = 0 \quad \forall g \qquad / \cdot \chi^\beta(g)^*, \sum_g$$

$$\Rightarrow 0 = \sum_\alpha (n^i_\alpha - n^j_\alpha) \sum_g \chi^\beta(g)^* \chi^\alpha(g) = \sum_\alpha (n^i_\alpha - n^j_\alpha)\, \#G\, \delta_{\alpha\beta}$$

$$= (n^i_\beta - n^j_\beta)\#G \Rightarrow n^i_\beta = n^j_\beta \quad \forall \beta$$

$\to \rho^i$ & ρ^j has the same decomposition to IRREPs

$\Rightarrow \rho^i \simeq \rho^j$ $\boxtimes$

Note: • we have prooved also

Theorem XX: Let (ρ, V) be reducible repre of a finite (complex Lie) G with decomposition to IRREPs

$$\rho = \bigoplus_\alpha n_\alpha \rho^\alpha$$

Then

$$n_\alpha = \frac{1}{\#G} \sum_g \chi^\alpha(g)^* \chi(g)$$

Note: • Th XX implies that the decomposition $\rho = \bigoplus_\alpha n_\alpha \rho^\alpha$ is unique

IRREDUCIBLE REPRESENTATIONS OF FINITE GROUPS

- remember: we know there is only a finite number of them: • #IRREPs ≤ number of distinct classes
  • $\sum_\mu d_\mu^2 \leq \# G$

Def: Regular representation of a finite G is

$$D^r(g_s)^k_\ell = \begin{cases} 1 & \text{for } g_s g_\ell = g_k \\ 0 & \text{otherwise} \end{cases}$$

- dim $D^r = \# G$
- in each row and each col. there is exactly one 1 (rearr.)
- it is indeed a repre:

  a) $g_s = e \Rightarrow 1$ for $g_\ell = g_k \Rightarrow D^r(e) = \mathbb{1}$

  b) $\sum_\ell D^r(g_r)^k_\ell D^r(g_s)^\ell_m = / g_r g_\ell = g_k \ \& \ g_s g_m = g_\ell$
  $\Rightarrow g_r g_s g_m = g_k / = D^r(g_r g_s)^k_m$ ✓

- character of D^r: • $\chi(e) = \# G$
  • $\chi(g \neq e) = 0$ (non-zero elements only off-diagonal: $g_s g_k = g_k \Rightarrow g_s = e$)

Example:

e	a	b
a	b	e
b	e	a

$$\Rightarrow D^r(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad D^r(b) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$D^r(a) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Theorem XXI: $\sum_\mu d_\mu^2 = \#G$ (sum over non-equiv. IRREPs) 42

Proof: • Maschke $\Rightarrow D^r$ is completely reducible

$$\Rightarrow D^r = \bigoplus_\mu n_\mu^r D^\mu$$

$$\Rightarrow n_\mu^r = \frac{1}{\#G} \sum_g \chi^\mu(g)^* \chi^r(g) = \frac{1}{\#G} \underbrace{\chi^\mu(e)^*}_{d_\mu} \chi^r(e) = d_\mu$$

$$\Rightarrow D^r = \bigoplus_\mu d_\mu D^\mu \Rightarrow \chi^r(g) = \sum_\mu d_\mu \chi^\mu(g) \quad / g = e$$

$$\Rightarrow \#G = \sum_\mu d_\mu^2 \qquad \boxtimes$$

• next we want to proove $\#IRREP_s = \#(\text{classes})$

Lemma:

Let C be a set of elements from G (each element can be incl. multiple - times).

Then $g C g^{-1} = C \quad \forall g \in G \Leftrightarrow C = \sum_{(g_k)} a_k (g_k)$.

• C need not be a group
• $\sum_{(g_k)}$ runs over distinct classes
• $a_k \geq 0 \Rightarrow$ elements of G might be included multiple-times in C
• note: $H \triangleleft G \Rightarrow H$ consists of complete classes.

Proof: $\Leftarrow$/ $g(g_k)g^{-1} = (g_k)$: $g(h g_k h^{-1})g^{-1} = \hat{h} g_k \hat{h}^{-1}$
(rearr.)

$\Rightarrow$/ • $g C g^{-1} = C$ & let $C = \sum_{(g_k)} a_k (g_k) + R$ such that $\exists a \in R : (a) \notin R$

• since $g(g_k)g^{-1} = (g_k) \; \forall (g_k) \; \forall g \in G \Rightarrow g R g^{-1} = R \; \forall g \in G$

$\Rightarrow gag^{-1} \in R \Rightarrow (a) \subset R$ ↯ $\boxtimes$

<u>Def</u>: (Multiplication of classes)

$$(g_k)(g_l) \equiv \{g_i g_j \mid g_i \in (g_k)\ g_j \in (g_l)\}$$ <u>incl. multiplicity</u>

<u>Example</u>: $C_{3v} = \{E\} + \{C_3, C_3^2\} + \{\sigma_v, \sigma_v', \sigma_v''\}$

$$(C_3)(C_3) = \{C_3^2, E, E, C_3\} = 2(E) + (C_3)$$
$$(C_3)(\sigma_v) = 2(\sigma_v)$$

<u>Def: (class const.)</u> $(g_i)(g_j) = \sum_{(g_k)} c_{ij}^k (g_k)$

- sum runs over all N_c distinct classes
- c_{ij}^k are <u>class constants</u>
- it is consistent:

$$g(g_i)(g_j)g^{-1} = g(g_i)g^{-1}g(g_j)g^{-1} = (g_i)(g_j) = /\text{lemma}/$$
$$= \sum_{(g_k)} a_k (g_k)$$

<u>Lemma</u>: Let (ρ^μ, V) be IRREP of a gr. G on a d_μ-dim vect. space. Then

$$m_i m_j \chi^\mu(g_i)\chi^\mu(g_j) = d_\mu \sum_{(g_k)} c_{ij}^k m_k \chi^\mu(g_k)$$

with $m_i = \#(g_i)$.

<u>Proof</u>: • $A_k = \sum_{h \in (g_k)} D^\mu(h)$ & $g(g_k) = (g_k)g$ (! not element-by-el.) $\Downarrow \sum_h$

$$\Rightarrow D^\mu(g) A_k = A_k D^\mu(g)$$

$\Rightarrow /SL2/ \Rightarrow A_k = \lambda \mathbb{1}$ for D^μ matrix IRREP

• $\mathrm{Tr}\, A_k = \lambda d_\mu = m_k \chi^\mu(g_k) \Rightarrow A_k = \frac{m_k}{d_\mu}\chi^\mu(g_k)\mathbb{1}$

- $(g_i)(g_j) = \sum_k C_{ij}^k (g_k) \Rightarrow A_i A_j = \sum_k C_{ij}^k A_k$

$$= \sum_k C_{ij}^k \frac{m_k}{d_\mu} \chi^\mu(g_k)\, \mathbb{1}$$

$$\Rightarrow Tr(A_i A_j) = \frac{m_i}{d_\mu}\frac{m_j}{d_\mu} \chi^\mu(g_i)\chi^\mu(g_j)\, \cancel{Tr(\mathbb{1})} = \sum_k C_{ij}^k \frac{m_k}{d_\mu}\chi^\mu(g_k)\, \cancel{Tr(\mathbb{1})}$$

$\boxtimes$

Theorem XXII: Number of non-equivalent IRREPs of a finite group is equal to the number of distinct classes.

Proof: • N_R – #IRREPs, N_c – number of classes; $m_k = \#(g_k)$

1, $\sum_{(g_k)} m_k \chi^\mu(g_k)^* \chi^\nu(g_k) = \#G\, \delta_{\mu\nu} \Rightarrow N_R \le N_c$

2, Lemma $\Rightarrow N_c \le N_R$:

- $m_i m_j \chi^\mu(g_i)\chi^\mu(g_j) = d_\mu \sum_{(g_k)} C_{ij}^k m_k \chi^\mu(g_k)$ for ρ^μ IRREP
- reg. repre $\Rightarrow \chi^r(g_k) = \delta_{k1}\#G = \sum_\alpha d_\alpha \chi^\alpha(g_k)$; $(g_1)=(e)$

$$\Rightarrow \underline{m_i m_j \sum_\mu \chi^\mu(g_i)\chi^\mu(g_j)} = \sum_{(g_k)} C_{ij}^k m_k \delta_{k1}\#G = \underline{C_{ij}^1 \#G}$$

- what is C_{ij}^1?

$$(g_i)(g_j) = C_{ij}^1 (e) + \sum_{k=2}^{N_c} C_{ij}^k (g_k)$$

$\Rightarrow C_{ij}^1(e) = m_i \delta_{ij'}$ where $(g_{j'})$ is the class of elements inverse to (g_j)

- $\underline{m_j = m_{j'}}$: $a \sim b \in (g_j)$, $a^{-1} \in (g_{j'}) \Rightarrow b = gag^{-1}$

$\Rightarrow b^{-1} = g a^{-1} g^{-1} \Rightarrow b \in (g_{j'})$

- $\chi^\mu(g_{j'}) = \chi^\mu(g_j)^*$ from equivalence with an unitary repre

$$\Rightarrow n_i n_j \sum_\mu \chi^\mu(g_i)^* \chi^\mu(g_j) = n_j \#G\, \delta_{ij}$$

$$\Rightarrow \sum_\mu \chi^\mu(g_i)^* \chi^\mu(g_j) \sim \delta_{ij} \qquad \Rightarrow N_C \leq N_R \quad \Rightarrow N_C = N_R \qquad \boxtimes$$

- orthogonality of N_C vectors of length N_R

<u>Theorem XXIII</u> (Frobenius irreducibility criterion)

Representation (ρ, V) of a finite group is irreducible

$$\Leftrightarrow \sum_g \chi(g)^* \chi(g) = \sum_{(g_u)} n_u\, \chi(g_u)^* \chi(g_u) = \#G$$

<u>Proof:</u> • $\rho = \bigoplus_\mu \alpha_\mu \rho^\mu \Rightarrow \quad \alpha_\mu = \frac{1}{\#G} \sum_g \chi^\mu(g)^* \chi(g) \in \mathbb{N}_0$!

$$\Rightarrow \chi(g) = \alpha_\mu \chi^\mu(g)$$

$$\Rightarrow \sum_g \chi(g)^* \chi(g) = \sum_{\mu\nu} \alpha_\mu \alpha_\nu \sum_g \chi^\mu(g)^* \chi^\nu(g) = \#G \sum_\mu \alpha_\mu^2$$

- for $\rho = \rho^\nu$ IRREP is $\alpha_\mu = \delta_{\mu\nu} \Rightarrow \sum_\mu \alpha_\mu^2 = 1$
- for ρ reducible is $\sum_\mu \alpha_\mu^2 > 1$ $\boxtimes$