RELATIONS BETWEEN REPRESENTATIONS OF A GROUP & ITS SUBGROUPS

A, SUBDUCED REPRESENTATIONS

Def: Let $T(g)$ be operators of a repre (ρ, V) of G and let $H < G$ is a subgroup. Then

$$\rho_{\downarrow H} = \{T(h) / h \in H\}$$

forms subduced representation of H

- $\rho_{\downarrow H}$ in general reducible even for ρ IRREP of G

$$\rho_{\downarrow H} = \bigoplus_{\mu} \alpha_{\mu}^{\downarrow H} \rho_{H}^{\mu} \iff \alpha_{\mu}^{\downarrow H} = \frac{1}{\# H} \sum_{h \in H} \chi_{H}^{\mu}(h)^{*} \chi(h)$$

Example: • $H = C_s = \{E, \sigma_v\} < G = C_{3v} = \{E, 2C_3, 3\sigma_v\}$

C_{3v}	E	$2C_3$	$3\sigma_v$
A_1	1	1	1
A_2	1	1	-1
E	2	-1	0

C_s	E	σ_v
A'	1	1
A''	1	-1

$\to A_{1\downarrow C_s} = A'$

$\to A_{2\downarrow C_s} = A''$

$\to E_{\downarrow C_s} = A' \oplus A''$

B, INDUCED REPRESENTATIONS (For finite groups)

- let $H < G$ & $D_H(h)$ is a d-dim. repre of H

$\Rightarrow$ can we construct a repre of the full G ?

YES, by explicit construction of its basis!

1, decomposition of G into left cosets with resp. to H

$$G = p_1 H + p_2 H + \dots + p_M H \qquad M = \frac{\#G}{\#H} \quad \text{(Lagrange)}$$

- p_i: fixed representatives of individual classes
- $p_1 = e$

2, basis of the induced repre of G

- let $\{\phi_1, \dots, \phi_d\}$ be the basis of D_H

$$\Rightarrow T(h)\phi_i = \sum_{j=1}^{d} \phi_j \, D_H(h)^j_{\ i}$$

$\mathrm{span}\{\phi_{ti}\} = \bigoplus_{t=1}^{M} p_t \underbrace{V_H}_{\text{isomorphic copy of } V_H}$ — repre space of D_H

- def. $\boxed{\phi_{ti} \equiv T(p_t)\phi_i \qquad t = 1, \dots, M;\ i = 1, \dots, d}$

(they are abstract objects not living in the repre space of D_H, we don't really know what they are...)

$\Rightarrow \phi_{ti}$ forms a basis of dM-dim repre of G

(we will not prove they are lin. indep.):

$$T(g)\phi_{ti} = T(g p_t)\phi_i = T(p_s p_s^{-1} g p_t)\phi_i = T(p_s) T(p_s^{-1} g p_t)\phi_i$$

- p_s: $p_s^{-1} g p_t \in H \Leftrightarrow g p_t \in p_s H$

(such p_s exists and is unique because $g p_t = g' \in G$ and each element of G belongs to exactly one coset)

$$\Rightarrow T(g)\phi_{ti} = T(p_s) \sum_j \phi_j D_H(p_s^{-1} g p_t)^j_{\ i} = \sum_j \phi_{sj} \, D_H(p_s^{-1} g p_t)^j_{\ i}$$

for $g p_t \in p_s H$

- def. $\boxed{\delta_{st}(g) = \begin{cases} 1 & g p_t \in p_s H \\ 0 & g p_t \notin p_s H \end{cases}}$

$$\Rightarrow \boxed{D_G(g)^{sj}_{\ ti} = \delta_{st}(g)\, D_H(p_s^{-1} g p_t)^j_{\ i}}$$

induced repre of G

$D_G = D_{H\uparrow G}$

- note that we dont really know what $T(p_t)\phi_i$ is but we don't need to in the end...

3, $D_G(g)^{sj}_{ti}$ is indeed a repre

a, $g = e \Rightarrow \delta_{st}(e) = \delta_{st}$: $ep_t = p_t \in p_t H$

$$\Rightarrow D_H(p_s^{-1} g p_t) = D_H(p_t^{-1} e p_t) = D_H(e) = \mathbb{1}$$

$$\Rightarrow D_G(e) = \mathbb{1}_{dM \times dM}$$

this selects one unique r from the $\sum_r$

b, $$\sum_{rk} D_G(g)^{sj}_{rk} D_G(g')^{rk}_{ti} = \sum_{rk} \delta_{sr}(g)\, \delta_{rt}(g')\, D_H(p_s^{-1} g p_r)^j_k D_H(p_r^{-1} g' p_t)^k_i$$

$= /$ $g p_r = p_s h$ & $g' p_t = p_r h' \Rightarrow g g' p_t = g p_r h' = p_s h h' \Leftrightarrow \delta_{st}(gg')$

& $\exists!\, p_r$ such that $g p_r \in p_s H \Rightarrow$ sum over r gives just single nonzero contrib. $/$

$$= \delta_{st}(gg')\, D_H(p_s^{-1} g g' p_t)^j_i \quad \boxtimes$$

<u>Theorem:</u> D_H unitary $\Rightarrow D_{G\uparrow H}$ unitary

<u>Proof:</u> $$[D_G(g)^{-1}]^{sj}_{ti} = [D_G(g^{-1})]^{sj}_{ti} = \delta_{st}(g^{-1})\, D_H(p_s^{-1} g^{-1} p_t)^j_i$$

$= /$ $g^{-1} p_t = p_s h \Leftrightarrow g p_s = p_t h' \Rightarrow \delta_{st}(g^{-1}) = \delta_{ts}(g)$ $/$

$$= \delta_{ts}(g)\, D_H((p_t^{-1} g p_s)^{-1})^j_i = /\text{unitarity}/ = \delta_{ts}(g) [D_H(p_t^{-1} g p_s)^i_j]^*$$

$$= [D_G(g)^+]^{sj}_{ti} \quad \boxtimes$$

- character of an induced repre

$$\chi_G(g) = \sum_{sj} \delta_{ss}(g) D_H(p_s^{-1} g\, p_s)^j_j = \sum_s \delta_{ss}(g)\, \chi_H(\underbrace{p_s^{-1} g\, p_s}_{\in H\,!})$$

- summation over M selected elements p_s can be replaced by a sum over $\forall g' \in G$ with additional condition $g'^{-1} g g' \in H \Rightarrow$ instead of a single representative of each coset we take every element from the coset; i.e., $\# H$ equal contributions instead of 1:

$g p_s \in p_s H$ $(\delta_{ss}(g))$ & $g' \in p_s H \Rightarrow g'^{-1} g\, g' = (p_s h')^{-1} g\, (p_s h)$
$= h'^{-1} p_s^{-1} g\, p_s h' = h'^{-1} h\, h' \in H \Rightarrow g g' \in g' H = p_s H$ ✓

$$\Rightarrow \boxed{\chi_G(g) = \sum_s \delta_{ss}(g)\, \chi_H(p_s^{-1} g\, p_s) = \frac{1}{\# H} \sum_{\substack{g' \\ g'^{-1} g g' \in H}} \chi_H(g'^{-1} g\, g')}$$

- both expressions useful – in different situations

- decomposition of $D_{H\uparrow G} = D_G$

– assume we are inducing IRREP of H: $\rho_G^{\nu\uparrow G}$ is a repre of G induced from the ρ_H^ν IRREP of H:

$$\rho_G^{\nu\uparrow G} = \bigoplus_\mu \alpha_\mu^{\nu\uparrow G} \rho_G^\mu \qquad\qquad \rho_H^{\mu\downarrow H} = \bigoplus_\nu \alpha_\nu^{\mu\downarrow H} \rho_H^\nu$$

Theorem XXV: (Frobenius)

$$\boxed{\alpha_\mu^{\nu\uparrow G} = \alpha_\nu^{\mu\downarrow H}}$$

in words: IRREP ρ_G^μ of G is contained in $\rho_G^{\nu\uparrow G}$ as many times as is the IRREP ρ_H^ν contained in $\rho_H^{\mu\downarrow H}$,

very easy to determine!

$$\Rightarrow \chi_G^{\nu\uparrow G}(g) = \sum_\mu \alpha_\mu^{\nu\uparrow G}\chi_G^\mu(g) = \sum_\mu \alpha_\nu^{\mu\downarrow G}\chi_G^\mu(g)$$

Proof: $$\alpha_\mu^{\nu\uparrow G} = \frac{1}{\#G}\sum_g \chi_G^\mu(g)^* \frac{1}{\#H}\sum_{\substack{g'\in G\\ g'^{-1}gg'\in H}} \chi_H^\nu(g'^{-1}gg') =$$

$$= \Big/ \begin{matrix} h = g'^{-1}gg' \Rightarrow g = g'hg'^{-1} \\ \Rightarrow \sum_g\sum_{g'} = \sum_h\sum_{g'} \end{matrix} \Big/ = \frac{1}{\#G\#H}\sum_{h\in H}\chi_H^\nu(h)\sum_{g'}\underbrace{\chi_G^\mu(g'hg'^{-1})^*}_{\#G\times\text{ the same!}}$$

$$= \frac{1}{\#H}\sum_{h\in H}\chi_H^\nu(h)\,\chi_G^\mu(h)^* = \left(\alpha_\nu^{\mu\downarrow H}\right)^* \underset{\underset{\mathbb{N}_0}{\Uparrow}}{=} \alpha_\nu^{\mu\downarrow H}$$

$\boxtimes$

Examples: 1, $H = \{e\}$, D_H is trivial repre $\Rightarrow$ $D_{H\uparrow G}$ is regular repre