SYMMETRIC (PERMUTATION) GROUP Sym(n)

Theorem (Cayley)

Every group G is isomorphic to a subgroup of the symmetric group $[Sym(G)]$ acting on G.

Note: ⇒ every row of the mult. table corresponds to some element of $Sym(G)$

1, elements of Sym(n), composition of permutations

$$P = \begin{pmatrix} 1 & 2 & \cdots & n \\ p_1 & p_2 & \cdots & p_n \end{pmatrix} \qquad S = \begin{pmatrix} 1 & 2 & \cdots & n \\ s_1 & s_2 & \cdots & s_n \end{pmatrix} = \begin{pmatrix} p_1 & p_2 & \cdots & p_n \\ s_{p_1} & s_{p_2} & \cdots & s_{p_n} \end{pmatrix}$$

↑ order is not relevant

$$\Rightarrow SP = \begin{pmatrix} 1 & 2 & \cdots & n \\ s_{p_1} & s_{p_2} & \cdots & s_{p_n} \end{pmatrix} \in Sym(n)$$

- $Sym(n)$ in general not Abelian
- $P^{-1} = \begin{pmatrix} p_1 & p_2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix} \qquad E = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}$

⇒ $Sym(n)$ is a group, $\# S_n = n!$

⇒ $Sym(m) < Sym(n) \quad \forall m < n$

2, Cycles

- cycle of length l is a permutation which leaves $n-l$ objects unchanged & l-objects are shifted without changing their order:

$$P = \begin{pmatrix} p_1 & p_2 & \cdots & p_l & p_{l+1} & \cdots & p_n \\ p_2 & p_2 & \cdots & p_1 & p_{l+1} & \cdots & p_n \end{pmatrix} = (p_1 p_2 \cdots p_l)$$

- the length l is the smallest exponent such that $P^l = E$
  ⇔ order of the cycle (element)
- transposition ≡ cycle of length 2
- any permutation can be decomposed to disjoint cycles:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 5 & 2 & 1 & 4 \end{pmatrix} = (1\,3\,5)(2\,6\,4)$$; · disjoint cycles commute

- disjoint (independent) cycles
  - don't have common element, commute
  - $\Rightarrow$ decomposition of P to dis. cycles is unambiguous up to order:

  1, take $a_1 \to a_2 \to \dots \to$ in a finite group $\exists a_l \to a_1 \Rightarrow$ cycle A_1

  2, take $b_1 \notin A_l \to \dots$ by repeating the same argument all elements are sorted to dis. cycles

- composition of cycles with common element

$$(abcd)(def) = \begin{pmatrix} a & b & c & d & e & f \\ b & c & d & a & e & f \end{pmatrix} \begin{pmatrix} a & b & c & d & e & f \\ a & b & c & e & f & d \end{pmatrix}$$

$$= \begin{pmatrix} a & b & c & e & f & d \\ b & c & d & e & f & a \end{pmatrix} \begin{pmatrix} a & b & c & d & e & f \\ a & b & c & e & f & d \end{pmatrix} = \begin{pmatrix} a & b & c & d & e & f \\ b & c & d & e & f & a \end{pmatrix}$$

$$\Rightarrow \boxed{(abcd)(def) = (abcdef)} \quad (*)$$

- composition of cycles with several common elements

$$(a_1 \dots a_i \, c \, a_{i+1} \dots a_j \, d)(d \, b_1 \dots b_r \, c \, b_{r+1} \dots b_s)$$

$$\overset{(*)}{=} (a_1 \dots a_i \, c)(c \, a_{i+1} \dots a_j \, d)(d \, b_1 \dots b_r \, c)(c \, b_{r+1} \dots b_s)$$

$$= (\qquad)(a_{i+1} \dots a_j \, d \, c)(c \, d \, b_1 \dots b_r)(\; - \text{"} - \;)$$

$$= (\; - \text{"} - \;)(a_{i+1} \dots a_j \, d)\underbrace{(dc)(cd)}_{E}(d \, b_1 \dots b_r)(\qquad)$$

$$= (a_1 \dots a_i \, c \, b_{r+1} \dots b_s)(a_{i+1} \dots a_j \, d \, b_1 \dots b_r)$$

3, Classes

- perms are conjugated $\Leftrightarrow$ they have identical cycle structure

$P = p_1 p_2 \dots p_m$ — p_i cycles

$Q = TPT^{-1} = Tp_1T^{-1}Tp_2T^{-1} \dots Tp_mT^{-1}$ $\Rightarrow$ Q is product of the same cycles in which the objects are permuted according to T:

- $T = (123) = (12)(23) \Rightarrow T^{-1} = (32)(21) = (321) = (132)$

$\Rightarrow T(2451)T^{-1} = (123)(4513) = (2345) = (3452)$ ✓ again cycle of order 4

- general: see page (5)

- number of elements in classes:

- class characterized by multiindex $\nu = (\nu_1, \nu_2, \dots, \nu_n)$

ν_i - number of cycles of order i

$$\Rightarrow (\nu) = (1^{\nu_1}, 2^{\nu_2}, \dots, n^{\nu_n})$$

$$\Rightarrow \#(\nu) = \frac{n!}{1^{\nu_1}\nu_1!\, 2^{\nu_2}\nu_2! \dots n^{\nu_n}\nu_n!}$$

$\nu_i! \leftrightarrow$ cycles can be permuted
$i^{\nu_i} \leftrightarrow$ objects within each cycle can cycled
} $\Rightarrow$ same element of S_n

Example $Sym(4) \Leftrightarrow 24$ elements

$\#(1^4) = \frac{4!}{1\ 4!} = 1$ — $(1)(2)(3)(4) = E$

$\#(1^2 2^1) = \frac{4!}{2!\ 2} = 6$ — $(12), (13), (14), (23), (24), (34)$

$\#(1^1 3^1) = \frac{4!}{1.1!\ 3} = 8$ — $(123), (124), (134), (234)$, $(132), (142), (143), (243)$

$\#(2^2) = \frac{4!}{2^2 2!} = 3$ — $(12)(34), (13)(24), (14)(23)$

$\#(4^1) = \frac{4!}{4} = 6$ — $(1234), (1324), (1243), (1423)$, $(1432), (1342)$

4, Cycle decomposition - transpositions

$$(a\,b\,c \dots p\,q) \overset{\circledast}{=} (ab)(bc)(c\cdot)\dots(\cdot p)(pq)$$

- this decomposition is not unique; however, for a given permutation, parity of the number of transpositions is unambiguous $\Rightarrow$ even vs. odd permutations

Proof: (Ma, p. 236) - Vandermond determinant

$$D(x_1, \dots, x_n) = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & & x_n \\ x_1^2 & x_2^2 & \dots & & x_n^2 \\ \vdots & \ddots & & & \\ x_1^{n-1} & \vdots & \dots & & x_n^{n-1} \end{vmatrix}$$

1, $\{x_i\} \to P\{x_i\}$ - D either changes sign or not

2, P decomposed to transp. & applied successively - D changes sign each time, final result must be the same

- subset of even permutations is normal subgroup of S_n (alternating group A_n)
  - it is kernel of the homomorf. $sgn: Sym(4) \to (\{1,-1\},\cdot)$
  - index of A_n in $Sym(n)$ is 2
    $Sym(n)/A_n \sim (\{1,-1\},\cdot)$
  - odd perms. are the only coset with resp. to A_n
  - $Sym(n) = A_n \circledS Sym(2)$
- decomposition to adjacent transpositions: $P_i = (i\ i{+}1)$

→ note: $P_i P_j = P_j P_i \quad \forall |i-j| \geq 2 \Rightarrow$ disjoint cycles

→ I want to prove

$$\boxed{(i\ i{+}k) = \prod_{l=i}^{i+k-1} P_l} \quad \text{for } k>1 \ \&\ i{+}k \leq n$$

a) $(i\ i{+}k) = (i\ i{+}k{-}1)(i{+}k{-}1\ i{+}k)(i\ i{+}k{-}1)$
(RHS is conjugation by the $(i\ i{+}k{-}1)$ transp.)

$RHS = (i\ i{+}k{-}1\ i{+}k)(i{+}k{-}1\ i) = (i{+}k\ i\ i{+}k{-}1)(i{+}k{-}1\ i)$
$= (i{+}k\ i)(i\ i{+}k{-}1)(i{+}k{-}1\ i) = (i{+}k\ i)$ ☒

b) repeat the decomposition: ξ

$= (i\ i{+}k{-}2)(i{+}k{-}2\ i{+}k{-}1)(i\ i{+}k{-}2)(i{+}k{-}1\ i{+}k)(i\ i{+}k{-}2)(i{+}k{-}1\ i{+}k{-}2)(i\ i{+}k{-}2)$

$= \ldots$ the decomposition expands only on both ends ... ⇒

$$\boxed{\begin{aligned}(i\ i{+}k) &= (i\ i{+}1)(i{+}1\ i{+}2)\cdots(i{+}k{-}1\ i{+}k)(i{+}k{-}1\ i{+}k{-}2)\cdots(i{+}1\ i) \\ &= (i\ i{+}1\ \ldots\ i{+}k)(i{+}k{-}1\ i{+}k{-}2\ \ldots\ i{+}1\ i)\end{aligned}}$$

5) Generators of $Sym(n)$

$P_1 = (1\ 2)$ & $W = (1\ 2\ \ldots\ n)$ generate whole $Sym(n)$:

- it is enough to show that $\boxed{P_{a+1} = W P_a W^{-1}}$ (+)
  → repeated conjugation of P_1 gives all adjacent transp.
  ⇒ any perm. can be composed from adj. transp.

Proof of (+)

- $W^n = 1 = W W^{n-1} \Rightarrow W^{n-1} = W^{-1}$ ✓
- W is shift of all objects by 1 position to the right:

$$W(i\ i{+}1)W^{-1} = W(i\ i{+}1)W^{n-1} = (i{+}1\ \ i{+}2) :$$

$$W = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 2 & 3 & \cdots & n & 1 \end{pmatrix} \Rightarrow W^{-1} = \begin{pmatrix} 2 & 3 & \cdots & n & 1 \\ 1 & 2 & \cdots & n-1 & n \end{pmatrix} = (1\ n\ n{-}1 \cdots 3\,2)$$

$$W(i\ i{+}1)W^{-1} = (1\ 2 \cdots i\ i{+}1 \cdots n{-}1\ n)(i\ i{+}1)(1\ n\ n{-}1 \cdots 3\,2)$$

$$= (i{+}2 \cdots n{-}1\ n\ 1\ 2 \cdots i)(i\ i{+}1)(i\ i{+}1)(1\ n\ n{-}1 \cdots 3\,2)$$

$$= (1\ 2 \cdots i{-}1\ i\ i{+}2 \cdots n)\underbrace{(i\ i{-}1 \cdots 1\ n\ n{-}1 \cdots i{+}2)}_{E}(i{+}2\ i{+}1) = (i{+}1\ i{+}2)$$ ☒

Representations of Sym(n)

[A.J. Coleman, The sym. group made easy; Adv. Q. Chem 4, 83 (1968)]

- classes (ν) characterized by decomposition $n = \sum_{i=1}^{n} i\nu_i$
- # IRREP $= N_c \Rightarrow$ can be characterized by analogous decomposition:

$$n = \sum_i \lambda_i, \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \Rightarrow \text{IRREP } [\lambda] = [\lambda_1, \lambda_2, \ldots, \lambda_n]$$

Note: • the relation $(\nu) \leftrightarrow [\lambda]$ is $\lambda_k = \sum_{j=k}^{n} \nu_j \quad k = 1, \ldots, n$

$$\Rightarrow \nu_i = \lambda_i - \lambda_{i+1}$$

- however, class (ν) & "corresp." IRREP $[\lambda]$ are in no direct relationship

1) Young diagrams – graphical representation of the decomp. $[\lambda]$

- it is n cells organized into rows of lengths $\lambda_1, \lambda_2, \ldots, \lambda_n$ satisfying $\lambda_i \geq \lambda_{i+1}$

Example: $Sym(3)$

 $\leftrightarrow [3] \leftrightarrow (1^3)$

 $\leftrightarrow [2,1] \leftrightarrow (1^1 2^1)$ $\Leftarrow \nu_i = \lambda_i - \lambda_{i+1}$

 $\leftrightarrow [1,1,1] \leftrightarrow (3^1)$

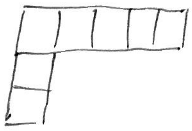
$\sim C_{3v}$
A_1
E
A_2

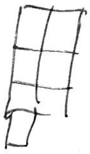
- standard ordering of YD:

$$[\lambda] > [\lambda'] \Leftrightarrow \exists j \mid \lambda_i = \lambda_i' \ \forall 1 \leq i < j \ \& \ \lambda_j > \lambda_j'$$

Example: $Sym(7)$

$[7], [6,1], [5,2], [5,1,1]^*, [4,3], [4,2,1], [4,1,1,1],$
$[3,2,2], [3,2,1,1], [3,1,1,1,1], [2,2,2,1]^{**}, [2,2,1,1,1],$
$[2,1,1,1,1,1], [7 \times 1]$

(*) $[5,1,1] =$  $\leftrightarrow (1^4 3^1)$

(**) $[2,2,2,1] =$  $\leftrightarrow (3^1 4^1)$

Note: • there is no closed formula for #IRREPs/classes of $Sym(n)$

2, Association of YD to IRREPs:

Def: Conjugate representations ρ & $\hat{\rho}$ are related by

$$\rho : P \mapsto D(P) \quad \Leftrightarrow \quad \hat{\rho} : P \mapsto sgn(P) D(P)$$

- it follows that if ρ is irreducible $\Rightarrow \hat{\rho}$ is irreducible:
- (in particular, ρ triv. IRREP $\Rightarrow \hat{\rho}$ is alternating (parity) IRREP)

$\hat{\chi}(P) = sgn(P)\, \chi(P)$: $\frac{1}{n!} \sum |\chi(P)|^2 = 1 \Rightarrow \frac{1}{n!} \sum |\hat{\chi}(P)|^2 = 1$ (Frobenius)

- $\frac{1}{n!} \sum \chi(P)^* sgn(P)\, \chi(P) \leq 1 \Rightarrow \hat{\rho}$ not equiv. unless $\chi(P) = 0$ for all odd permutations (then $\rho \sim \hat{\rho}$)

- YD of conjugate representations are related by transpositions

1, • • • • ⟷ triv ⟺ ⋮ parity

4, ⁚˙ $\xrightarrow{\text{trans}}$ ⁚˙ ⟹ ρ self-conjugate, $\chi(P) = 0$ for P odd

3, Young tableaux

- each cell in YD is filled by numbers $1, 2, \dots, n$ such that the numbers increase in each row from left to right & in each col from top down
- number of distinct legal YT determines the dimension of the representation ⟹ YT index basis vecs of the IRREP

Example: $Sym(3)$ ⟹ $\# G = 3! = 6$

1, • • • is triv. repre ⟹ 1-dim ρ_1

2, ⁚˙ ⟶ $\dim \rho_2 = ?$

3, ⋮ parity $\hat{\rho}_1$ ⟹ 1-dim

$\sum d_\mu^2 = 6 \Rightarrow \dim$ ⁚˙ $= 2$

YT: • • • ⟹ only 1 2 3

⁚˙ ⟹ $\begin{matrix} 1 & 2 \\ 3 & \end{matrix}$ & $\begin{matrix} 1 & 3 \\ 2 & \end{matrix}$ ✓

⋮ ⟹ only $\begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$

- standard ordering of YT:
  - ⟶ compare numbers in first row from left to right, then in 2nd row ... ⟹ first different number determines the order (lower number ⟺ lower YT)

$$\begin{matrix} 1 & 2 \\ 3 & \end{matrix} < \begin{matrix} 1 & 3 \\ 2 & \end{matrix}$$

4) dimensions of IRREPs – hook rule

- hook number: for the (i,j) cell of YD, $h_{ij} = 1 + \#$(cells in ith row to the right) & $\#$(cells in j-th col down)

$h_{ij} \equiv$ hook length (of the hook with (i,j) being head node)

- hook table: YD filled with hook numbers

Example: • Sym(4), $\rho_{[2,1,1]}$

4	1
2	
1	

$$\Rightarrow d_{[2,1,1]} = \frac{4!}{4 \cdot 2} = 3$$

- dimension of $\rho_{[\lambda]}$: $d_{[\lambda]} = \frac{n!}{\prod_{ij} h_{ij}}$ $\Leftrightarrow$ number of YT's

YT:

```
1 4    1 3    1 2
2      2      3
3      4      4
```

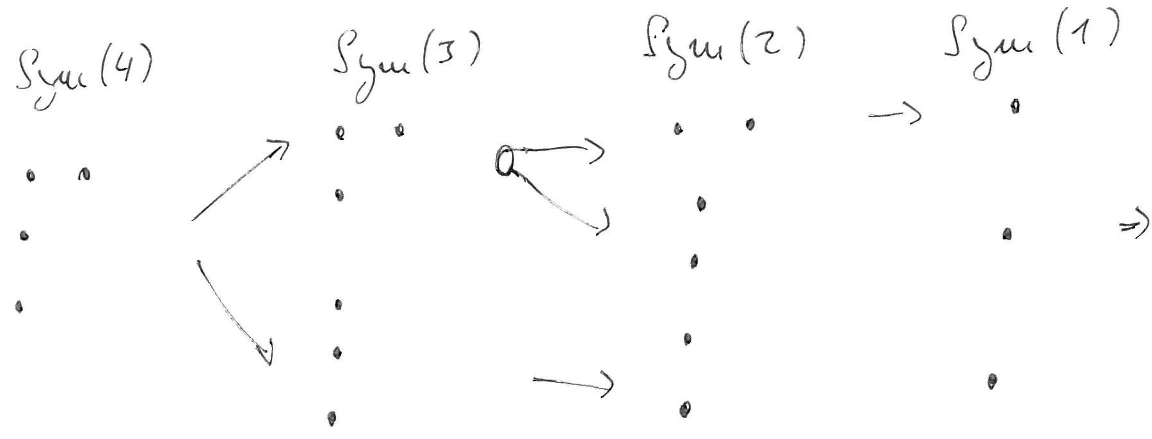
5) Why and how it works?

- determination of $d_{[\lambda]}$ & indexing of the basis vectors by YT is based on the decomposition of subduced representations in the hierarchy

$$\mathrm{Sym}(n) \downarrow \mathrm{Sym}(n-1) \downarrow \dots \downarrow \mathrm{Sym}(1)$$

Theorem: Repre of Sym(n-1) subduced from IRREP $\rho_{[\lambda]}$ of Sym(n) is direct sum of IRREPs of Sym(n-1) corresponding to YD's obtained from $[\lambda]$ by detachment of one cell such that the resulting diagram is again legal YD.

Example: $\rho_{[2,1,1]}$ of Sym(4)



$\Rightarrow \dim \rho_{[2,1,1]} = 3$ ✓

- relation to YT:
  - looking at the 3 YT of $\vdots\,\,\dot{}$, we note that in the first subduction ($S_4 \downarrow S_3'$) we detach all cells that can contain 4, in $S_3 \downarrow S_2$ subd. all cells that can contain 3 and so on

  $\Rightarrow$ YT correspond to individual branches of the subduction tree

6, Characters of IRREPs of $Sym(n)$ ... $\chi^{[\lambda]}$

(practical algorithm using hook table)

a, first column of HT: lengths of main hooks $h_{i1} \equiv h_i$

b, define symbol $D = |h_1\ h_2\ \cdots\ h_r|$ $\quad r$ ... # of rows of YD

c, rules for evaluating D:

i, $D = 0$ if $\exists i: h_i < 0$ or if $\exists i \neq j: h_i = h_j$

ii, D changes sign upon exchanging h_i & h_j

iii, $D_0 \equiv |r-1\ \ r-2\ \cdots\ 1\ 0| = 1$

iv, $\mu D = |h_1 - \mu\ \ h_2 \cdots h_r| + |h_1\ \ h_2 - \mu \cdots h_r| + \dots + |h_1\ h_2 \cdots h_r - \mu|$

d, $\chi^{[\lambda]}(\nu) = \chi^{[\lambda_1, \dots, \lambda_n]}(1^{\nu_1}, 2^{\nu_2}, \dots, n^{\nu_n})$ is determined by successively multiplicating D ν^n-times by n, ν^{n-1} times by $n-1$, ..., and ν_1-times by 1.

Example:

	E	O	E	E	O	
$Sym(4)$	$1(1^4)$	$6(1^2 2)$	$3(2^2)$	$8(1^1 3^1)$	$6(4^1)$	
[4]	1	1	1	1	1	← triv
[3,1]	3	1	−1	0	−1	
[2,2]	2	0	2	−1	0	← self-conj.
[2,1,1]	3	−1	−1	0	1	
[1,1,1,1]	1	−1	1	1	−1	← parity

- dimensions:

$[4]$: 4 3 2 1 $\Rightarrow d_{[4]} = 1$

$[3,1]$: 4 2 1 / 1 $\Rightarrow d_{[3,1]} = 3$ $\xrightarrow{\text{conj.}}$ $[2,1,1]$: 4 1 / 2 / 1 $\Rightarrow d_{[2,1,1]} = 3$

$[2,2]$: 3 2 / 2 1 $\Rightarrow d_{[2,2]} = 2$

- characters:

$\chi^{[3,1]}(1^2, 2)$: $D = |4 \; 1|$

- $2D = |2 \; 1| + |4 \; -1| = |2 \; 1|$
- $1/|2 \; 1| = |1 \; 1| + |2 \; 0| = |2 \; 0|$
- $1/|2 \; 0| = |1 \; 0| = 1$

$\Rightarrow \chi^{[3,1]}(1^2, 2) = 1$

$\chi^{[2,2]}(2^2)$: $D = |3 \; 2|$

$2D = |1 \; 2| + |3 \; 0| = |3 \; 0| - |2 \; 1|$

$2/|3 \; 0| - 2/|2 \; 1| = |1 \; 0| - |0 \; 1| = |1 \; 0| + |1 \; 0| = 2$

- rules (orthogonalities etc) still apply

(Ortogonální) Maticové reprezentace S_n

$D^{[\lambda]}(i\ i+1)$ lze zkonstruovat následovně:

1, std. YT pro $[\lambda]$ ve vzestupném pořadí

Př: S_4, $[\lambda] = [3,1]$, $d_{[\lambda]} = 3$

(1) $\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}$ (2) $\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}$ (3) $\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}$

- i, $j = i+1$ najdu v každé tabulce:
  - a, ve stejném řádku
  - b, ve stejném sloupci
  - c, v různých řádcích a sloupcích, ale prohozením $i \leftrightarrow j$ dostaneme opět YT

2, řádky a sloupce matic D číslujeme Y tabulkami (vzestupně) a maticové elementy určíme následovně:

i, pro YT typu (a) napíšeme 1 na diag. pozici a 0 jinde v příslušném řádku a sloupci

ii, pro YT (b) -1 na diag. pozici, 0 jinde

iii, elementy odpovídající dvěma YT typu (c) vyplníme blokem $\begin{pmatrix} -\rho & \sqrt{1-\rho^2} \\ \sqrt{1-\rho^2} & \rho \end{pmatrix}$ a nulami jinde v odp. řádcích a sloupcích

- ρ^{-1} je počet kroků podél „háku" $i \to j$

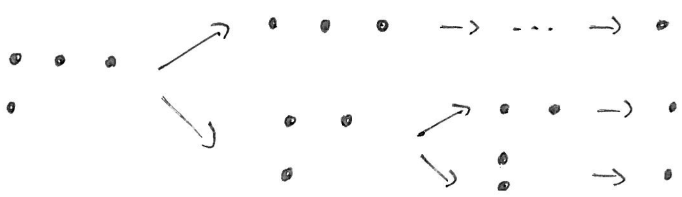
$$D^{[3,1]}(12) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad D^{[3,1]}(23) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/2 & \frac{\sqrt{3}}{2} \\ 0 & \sqrt{3}/2 & 1/2 \end{pmatrix}$$

$$D^{[3,1]}(34) = \begin{pmatrix} -1/3 & \frac{2\sqrt{2}}{3} & 0 \\ \frac{2\sqrt{2}}{3} & 1/3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- ostatní permutace násobením těchto základních transpozic

NB: Subdukce $S_n \to S_{n-1}$ $\Leftrightarrow$ vynechání $(n-1\ n)$

$\Rightarrow$ vidíme, že dostáváme přímo reducibilitu $[3] \oplus [2,1]$



Eliot, Dawber sec. 8.6.4

Dodatky: symetrie vlnové funkce tří elektronů

$\psi(\vec{r}_i, \sigma_i) = \phi(\vec{r}_i) \cdot \chi(\sigma_i)$

- celkem musí být úplně antisymetrická $\Leftrightarrow \psi \sim a$
- $S_3 \sim D_3$

S_3	even E	even (123) (132)	odd $(12),(23),(13)$	
s	1	1	1	(sym)
a	1	1	-1	(anti-sym)
m	2	-1	0	(mixed)
$s\otimes s$	1	1	+1	= s
$s\otimes a$	1	1	-1	= a
$a\otimes a$	1	1	1	= s
$m\otimes a$	2	-1	0	= m
$m\otimes s$	2	-1	0	= m
$m\otimes m$	4	1	0	$= m\oplus s\oplus a$

$\Rightarrow$ povolené symetrie $\phi\cdot\chi$ jsou $s\otimes a$, $a\otimes s$ a $m\otimes m$

- spinová část: $(s=\tfrac12)\otimes(\tfrac12)\otimes(\tfrac12) = (1)\otimes(\tfrac12) \,\&\, (0)\times(\tfrac12)$
  $= (\tfrac32) \,\&\, (\tfrac12)_1 \,\&\, (\tfrac12)_0$
- $(\tfrac32)$ jsou sym, neboť $(j=\tfrac32, m=\tfrac32)$ je symetrický a $J_\pm$ je symetrický operátor ($J_\pm = J_1 \pm i J_2$, $J_1 = \sum s_1$, $J_2 = \sum s_2$)
  - jsou 4 symetrické spinové stavy
- antisym. stavy nejsou: 3 částice rozmisťujeme do dvou jednočást. stavů $\Rightarrow$ min. dvě musí být ve stejném

$\Rightarrow$ oba $(\tfrac12)_1$ a $(\tfrac12)_0$ stavy jsou smíšené

Spin: 4s, 4m
orb: 1a, 10s, 16m
$(s)\times(a) \to 4\ (a)$ stavy ^4S
$(m)\times(m) = 4\times 16$ stavů, z toho 16 (a)
$\Rightarrow {}^2P, {}^2D$ ✓

- Př: N $\Rightarrow$ 3 valenční elektrony jsou $2p \Leftrightarrow l=1$
  $\Rightarrow$ celkem $(l=1)^3 = 27$ stavů $= \underset{0}{S} + 3\underset{1}{P} + 2\underset{2}{D} + \underset{3}{F}$
  - $(l=3, m=3)$ je sym $\Rightarrow$ F dává 7 sym. stavů
  - celkem je $\frac16 n(n+1)(n+2) = 10$ sym. stavů $(n=3)$ $\Rightarrow$ jeden z P musí být také sym.
  - S je antisym.(*), celkem je $\frac16 n(n-1)(n-2) = 1$ antisym. stav $\Rightarrow$ 2P, 2D je mix
  $\Rightarrow$ dovolené stavy jsou ^4S, ^2P, ^2D. $(\# = 20)$

(*) je to $|\phi_1(m=-1)\,\phi_2(m=0)\,\phi_3(m=1)|$ determinant