CHARACTER TABLE OF A POINT GROUP

- goal: find characters of all IRREPs of a given groups
  $\Leftrightarrow$ for each IRREP find $\chi^{\mu}(g_u)$ for all distinct classes:

G	(e)	(g_2)	$\cdots$	(g_{N_c})
ρ^1				
ρ_2				
$\vdots$				
ρ_{N_c}				

- it can be done in a few steps without explicit construction of the matrices:

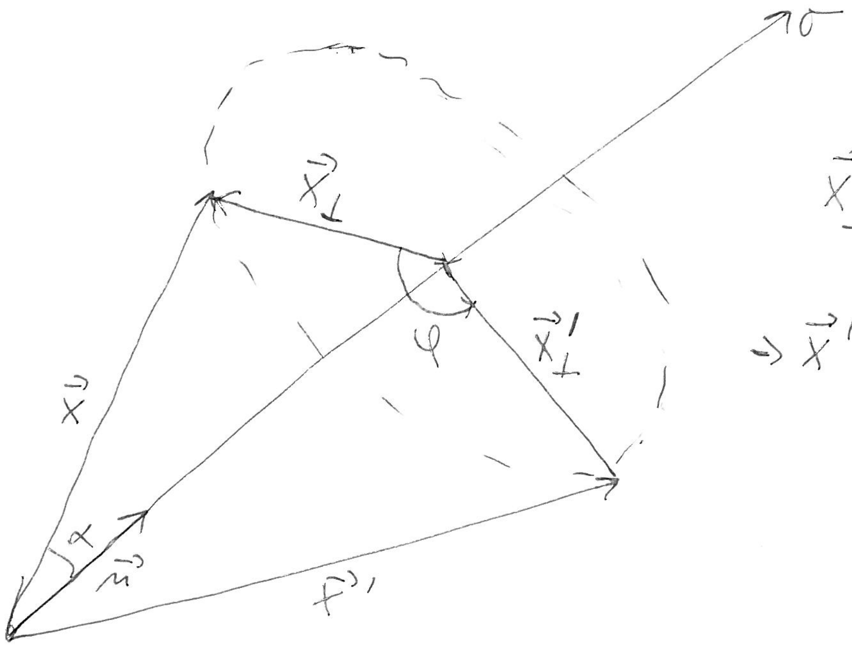
1, $N_R = N_c$ (already used)
2, $\sum d_\mu^2 = \# G \Rightarrow$ first col. $\chi^\mu(e)$
3, trivi. representation (scalar) $\Rightarrow$ first row
   3b, might exist pseudoscalar
4, decomposition of a vector (& pseudo vec.) repre
5, orthogonality of columns:
$$\sum_\alpha \chi^\alpha(g_u)^* \chi^\alpha(g_e) = \frac{\# G}{m_u} \delta_{ue}$$
6, orthogonality of rows:
$$\sum_{(g_u)} m_u \chi^\alpha(g_u)^* \chi^\beta(g_u) = \# G \delta_{\alpha\beta}$$
$$\Downarrow$$
7, Frobenius (for rows) $\alpha = \beta$
8, $\chi(g_1 g_2) = \chi(g_1)\chi(g_2)$ for 1D-repre

VECTOR & PSEUDO-VECTOR REPRESENTATIONS OF $O(3)$

$\Leftrightarrow$ action of $O(3)$ on $\mathbb{R}^3$ $\rightarrow$ 3D-repre

1) rotation of a vector $\vec{x}$ around $\vec{n}$ by an angle φ:

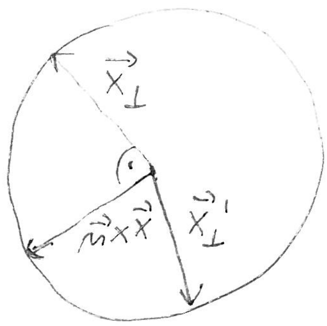
$$\vec{x}\,' = D(C_\varphi^{\vec{n}})\,\vec{x}$$



$$\vec{x}_\perp = \vec{x} - (\vec{n}\cdot\vec{x})\vec{x}$$

$$\Rightarrow \vec{x}\,' = (\vec{n}\cdot\vec{x})\vec{n} + \vec{x}_\perp'$$

- basis in a plane $\perp \vec{n}$: $\vec{x}_\perp$ & $\vec{n}\times\vec{x}$



$$\Rightarrow |\vec{x}_\perp| = |\vec{n}\times\vec{x}| = |\vec{x}|\sin\alpha$$

$$\Rightarrow \vec{x}_\perp' = \vec{x}_\perp\cos\varphi + (\vec{n}\times\vec{x})\sin\varphi$$

$$\Rightarrow \vec{x}\,' = (\vec{n}\cdot\vec{x})\vec{n} + (\vec{x} - (\vec{n}\cdot\vec{x})\vec{n})\cos\varphi + (\vec{n}\times\vec{x})\sin\varphi$$

$$\Rightarrow \boxed{\vec{x}\,' = \vec{x}\cdot\cos\varphi + (1-\cos\varphi)(\vec{n}\cdot\vec{x})\vec{n} - (\vec{x}\times\vec{n})\sin\varphi}$$

$$(\vec{x}\times\vec{n})_i = \varepsilon_{ijk}\,x_j\,n_k$$

$$\Rightarrow x_i' = x_i\cos\varphi + (1-\cos\varphi)\,n_i n_j x_j - \sin\varphi\,\varepsilon_{ijk}\,n_k x_j$$

$$\Rightarrow \boxed{D(C_\varphi^{\vec{n}})_{ij} = \delta_{ij}\cos\varphi + (1-\cos\varphi)\,n_i n_j - \sin\varphi\,\varepsilon_{ijk}\,n_k}$$

2, improper rotation (rot. + reflection through parallel plane) (T2.2)

$$D(S_\varphi^{\vec{n}})\vec{x} = D(C_\varphi^{\vec{n}})\vec{x} - 2(\vec{n}\cdot\vec{x})\vec{n}$$

$$\Rightarrow \boxed{D(S_\varphi^{\vec{n}})_{ij} = \delta_{ij}\cos\varphi - (1+\cos\varphi)n_i n_j - \sin\varphi\, \varepsilon_{ijk} n_k}$$

$\Rightarrow$ characters of the vector representation

$\sum n_i^2 = 1$

$$\Rightarrow \boxed{\begin{aligned} \chi^v(C_\varphi^{\vec{n}}) &= 3\cos\varphi + (1-\cos\varphi) = 1+2\cos\varphi \\ \chi^v(S_\varphi^{\vec{n}}) &= 3\cos\varphi - (1+\cos\varphi) = -1+2\cos\varphi \end{aligned}}$$

Note: • $\chi(C_\varphi^{\vec{n}})$ & $\chi(S_\varphi^{\vec{n}})$ are indep. of $\vec{n}$

$\Rightarrow$ indication that $C_\varphi^{\vec{n}}$ for any $\vec{n}$ form a class (same for $S_\varphi^{\vec{n}}$)

$\Rightarrow$ it is indeed the case as we can transform any axis to any other using some rotation from $O(3)$

• for $O(3)$, vec. repre is irreducible

• for point groups $\subset O(3)$ this need not be the case

3) pseudovector representation : $\vec{R} = \vec{x} \times \vec{y}$

$$\Rightarrow D^p(g) = (\det D^v(g))\, D^v(g) \quad \ldots \text{ exercice}$$

$$\Rightarrow \chi^p(C_\varphi^{\vec{n}}) = 1+2\cos\varphi \qquad \Leftarrow \det D^v(C) = 1$$

$$\chi^p(S_\varphi^{\vec{n}}) = 1-2\cos\varphi \qquad \Leftarrow \det D^v(S) = -1$$