LIE GROUPS

Introduction & worked example

- continuous group – infinite
  - elements are continuously parametrized by n real parameters:

$$g(x_1, \dots, x_n) \in G \qquad \dim G = n$$

- group operation is described by a function
  - $g(x)g(y) = g(z)$

$$z_i = \varphi_i(x_1, \dots, x_n; y_1, \dots, y_n)$$

  - $g^{-1}(x) = g(y)$

$$y_i = c_i(x_1, \dots, x_n)$$

- if φ_i & c_i are C^∞, we speak of Lie group

Note: • Lie group is a differentiable manifold
$\Rightarrow$ the functions φ_i, c_i are defined locally for each chart domain + smooth $\phi_i \circ \phi_j^{-1}$ on intersection of domains

- special (and in fact nearly the only relevant) class are matrix groups – n^2 matrix elements param. by n parameters

Example: 1, $SO(3)$ ... group of rotations in $\mathbb{R}^3$

- $SO(3)$ ~ group of ~~or~~thogonal 3×3 matrices with $\det = 1$:

a, $R \in \mathcal{M}_{3\times3}(\mathbb{R})$ • non-Abelian group

b, $R^T R = \mathbb{1}_3$

c, $\det R = 1$

- $\dim G = 3$ (b, consists of 6 conditions; c, does not bring anything more)
- param: Euler angles

- it is extremely useful to study Lie groups by means of linearization in the neighborhood of e
  - $\Rightarrow$ leads to Lie algebra (vector space with anti-sym. bilinear form - commutator)
  - $\Rightarrow$ much simpler object
- why:
  - are two Lie groups isomorphic?
    - a, the two manifolds must be homeomorphic (deformable to each other)
    - b, the composition & inversion functions (both nonlinear) must be equivalent
  - $\Rightarrow$ very difficult questions, at the linearized level of LA much easier
  - $\Rightarrow$ something is lost, of course)

- Marius Sophus Lie (1842 - 1899) - Norwegian physicist

- LA of $SO(3)$ - $so(3)$
  - $R(\epsilon\vec{\varphi}) \simeq \mathbb{1}_3 + A \qquad A = O(\epsilon)$
  - $R^T R = (\mathbb{1} + A)(\mathbb{1} + A^T) = \mathbb{1} + A + A^T + O(\epsilon^2) = \mathbb{1}$

    $\Rightarrow \boxed{A^T = A} \Rightarrow A$ is anti-symmetric matrix
  - basis of anti-sym matrices 3×3 $\equiv$ generators of infinitezimal transformations

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad J_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow R \simeq \mathbb{1} + \epsilon \varphi_i J_i + O(\epsilon)$ in the neighborhood of $\mathbb{1}$

NOTE: • in physics, generators often defined as hermitian

$\Rightarrow J_\alpha \to \pm i J_\alpha$

- finite rotations:

$$R(\vec{\varphi}) = \lim_{N\to\infty} R\left(\frac{\vec{\varphi}}{N}\right)^N = \lim\left(\mathbb{1} + \frac{\varphi_i J_i}{N}\right)^N \to \exp(\varphi_i J_i)$$

⇒ from the generators, finite-angle transformation is obtained through exponential mapping

- det $R(\vec{\varphi}) = 1$?
  - → identity $\det R = \exp(\mathrm{Tr}\log R)$
    ($\Leftarrow R = S^{-1}\Lambda S$, $\Lambda = (\lambda_1, \dots, \lambda_n)$ & sum for exp)
  - → $R \simeq \mathbb{1} + A \Rightarrow \log R \simeq A \Rightarrow \mathrm{Tr}\log R = 0 \Rightarrow \det R = 1$
  - ⇒ $\det R = 1$ is a consequence of $R^T R = \mathbb{1}$; or is it?

- group $O(3) > SO(3)$:
  - → $R^T R = \mathbb{1}$ is the only condition
  - ⇒ $\det(R^T R) = 1 = (\det R)^2 \Rightarrow \det R = \pm 1$
  - → $A^T = -A$ is consequence of $R^T R = \mathbb{1} \Rightarrow O(3)$ has the same LA as $SO(3)$
  - ⇒ exp does not recover the whole group
  - → only the connected subgroup is accessible from LA via exp; for $O(3)$ it is $SO(3)$

Lie algebra:

- G Lie group, $\mathcal{G}$ it's LA:
$$g = \mathbb{1} + \epsilon A \qquad g, g' \in U(e) \subset G$$
$$g' = \mathbb{1} + \epsilon B \qquad A, B \in \mathcal{G}$$

1, $\mathbb{1} + \epsilon A \in G$ ($\alpha \in \mathbb{R}$) $\Rightarrow \mathbb{1} + \alpha\epsilon A \in G \Rightarrow A \in \mathcal{G} \Rightarrow \alpha A \in \mathcal{G}$

2, $gg' \in G \Rightarrow (\mathbb{1} + \epsilon A)(\mathbb{1} + \epsilon B) = \mathbb{1} + \epsilon(A+B) \in G$

→ $A + B \in \mathcal{G}$ → $\mathcal{G}$ is vector space

5, $gg'g^{-1}g'^{-1} \in G$ ... group "commutator"

- $g = e^{\epsilon A} \sim \mathbb{1} + \epsilon A \qquad g' = e^{\delta B} \sim \mathbb{1} + \delta B$
- Baker-Campbell-Hausdorff

$$e^X \cdot e^Y = e^{X + Y + \frac{1}{2}[X,Y] + \dots}$$

$$\Rightarrow gg'g^{-1}g'^{-1} = e^{\epsilon A + \delta B + \frac{\epsilon\delta}{2}[A,B]} e^{-\epsilon A} e^{-\delta B}$$

$$= e^{\delta B + \frac{\epsilon\delta}{2}[A,B] - \frac{\epsilon\delta}{2}[B,A]} e^{-\delta B}$$

$$= e^{\epsilon\delta[A,B]} \sim \mathbb{1} + \epsilon\delta[A,B] \quad \Rightarrow [A,B] \in \mathcal{G}$$

$\Rightarrow$ LA $\mathcal{G}$ is closed under commutator

Example: $\cdot\ A^T = -A,\ B^T = -B \Rightarrow [A,B]^T = -[A,B]$:

$$(AB)^T = B^T A^T \Rightarrow [A,B]^T = (AB)^T - (BA)^T = B^T A^T - A^T B^T$$

$$= +BA - AB = [B,A] = -[A,B]$$

$\Rightarrow so(3)$ is closed under matrix commutator

Def: Real Lie algebra $\mathcal{L}$ is a real vector space with a bilin. operation (commutator) $[\cdot,\cdot]$:

$\forall A,B,C \in \mathcal{L}$, $\forall \alpha,\beta \in \mathbb{R}$:

1, $A,B \in \mathcal{L} \Rightarrow [A,B] \in \mathcal{L}$ (closure)

2, $[\alpha A + \beta B, C] = \alpha[A,C] + \beta[B,C]$ (lin)

3, $[A,B] = -[B,C]$ (anti/skew-symmetry)

4, $[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$ (Jacobi)

Notes: $\cdot$ 2, + 3, $\Rightarrow$ bilinearity

- 4, is connected with the integrability of the function φ of group operation

Examples:

- anti-sym matrices
- all $n\times n$ matrices
- functions on a phase space with Poisson bracket

Def: <u>Structure constants of LA $\mathcal{L}$</u>

Let $\{J_i\}_{i=1}^{n}$ be generators (basis) of LA $\mathcal{L}$. Then

$$[J_i, J_j] = C_{ij}^{k} J_k \in \mathcal{L}$$

and C_{ij}^{k} are called <u>structure constants of $\mathcal{L}$</u>.

<u>Notes</u>:
- C_{ij}^{k} depend on the choice of basis, but for a given basis fully define $[\cdot,\cdot]$ & the structure of LA
  $A = A^i J_i$, $B = B^j J_j$ $\Rightarrow$ $[A, B] = A^i B^j [J_i, J_j] = A^i B^j C_{ij}^{k} J_k$
- $C_{ji}^{k} = - C_{ij}^{k}$ (skew-sym.)
- $C_{pq}^{s} C_{rs}^{t} + C_{qr}^{s} C_{ps}^{t} + C_{rp}^{s} C_{qs}^{t} = 0$ (Jacobi)

<u>Example</u>: · $so(3)$ $\Rightarrow$ $C_{ij}^{k} = \varepsilon_{ijk}$

→ hermitian generators $J_i \to \pm i J_i$ $\Rightarrow$ $C_{ij}^{k} = i\varepsilon_{ijk}$

$\Rightarrow$ commut. relations of orbital impulsmoment

<u>Conclusions</u>:
- LA reflects algebraic structure of the LG G through commutator in the neighborhood of identity
  $g g' g^{-1} g'^{-1} \to [A, B]$
- LA is a vector space $\Rightarrow$ inner product can be defined; with proper choice (Cartan-Killing form), LA reflects also topological properties of G
- from topol. point of view, all points on the manifold are equivalent ($g = eg$) $\Rightarrow$ we could linearize around any g; algebraically, e is special
- instead of group representations, we can construct representations of LA $\Rightarrow$ exp

Note: adjoint (regular) repre of LA

- for $so(3)$, $(J_i)^j_{\ k} = -\varepsilon_{ijk} = -C_{ij}^{\ \ k} = \varepsilon_{ikj} = C_{ik}^{\ \ j}$

- $(T_a)^b_{\ c} = C_{a\ c}^{\ b}$ is representation of any abstract algebra with str. consts. $C_{a\ c}^{\ b}$:

$$[T_a,[T_b,T_c]] = C_{bc}^{\ \ k}[T_a,T_k] = C_{bc}^{\ \ k} C_{ak}^{\ \ \ell} T_\ell \quad \left(\begin{smallmatrix}\text{not}\\ \text{needed}\end{smallmatrix}\right)$$

$\Rightarrow$ Jacobi identity:

$$C_{bc}^{\ \ k} C_{ak}^{\ \ \ell} + C_{ca}^{\ \ k} C_{bk}^{\ \ \ell} + C_{ab}^{\ \ k} C_{ck}^{\ \ \ell} = 0$$

$$(T_b)^k_{\ c}(T_a)^\ell_{\ k} - (T_a)^k_{\ c}(T_b)^\ell_{\ k} - C_{ab}^{\ \ k}(T_k)^\ell_{\ c}$$

$$\Rightarrow C_{ab}^{\ \ k}(T_k)^\ell_{\ c} = (T_aT_b - T_bT_a)^\ell_{\ c} - [T_a,T_b]^\ell_{\ c}$$

$\Rightarrow (T_a)^b_{\ c} = C_{ac}^{\ \ b}$ is repre of LA with str. constants $C_{ac}^{\ \ b}$

P. Picasso

To arrive at abstraction, it is always necessary to begin with a concrete reality ... You must always start with something. Afterward you can remove all traces of reality

Key concepts of differential geometry

Def: Topological space (X, τ) is a set X with collection τ of open subsets (topology), which satisfy:

1, $X \in \tau$, $\emptyset \in \tau$

2, $\bigcup_{\alpha \in A} \tau_\alpha \in \tau \quad \forall \tau_\alpha \in \tau$ & A arb. set of indices (also uncountable)

3, $\bigcap_{i=1}^{n} \tau_i \in \tau \quad \forall \tau_i \in \tau$, $n < +\infty$ (finite intersection)

- closed subset: $A = X \setminus \tau_i$ ... complement of an open subs.
  - $X, \emptyset$ both closed & open

Def: Neighborhood of a point $x \in (X, \tau)$ is a set $\mathcal{U}(x) \subset X$ that includes an open subset containing x:

$\mathcal{U}(x)$ is neighborhood $\Rightarrow \exists\, \sigma \in \tau : x \in \sigma$ & $\sigma \subset \mathcal{U}(x)$

Def: Mapping $\phi : (X_1, \tau_1) \to (X_2, \tau_2)$ is

a, continuous in a point $x \in (X_1, \tau_1)$, if
$\forall\, \mathcal{U}_2(\phi(x)) \subset X_2 \ \exists\, \mathcal{U}_1(x) \in X_1 : \phi(\mathcal{U}_1(x)) \subset \mathcal{U}_2(\phi(x))$

b, continuous if $\forall\, \sigma \in \tau_2 \quad \phi^{-1}(\sigma) \in \tau_1$
(inverse image of every open set in X_2 is open in X_1)

Note:
- $\phi^{-1}(A) = \{x \in X_1 : \phi(x) \in A\}$ does not require existence of an inverse mapping
- continuity $\Leftrightarrow$ continuity in every point

Def: Top. space is connected if it can't be divided into two disjoint non-empty open sets.

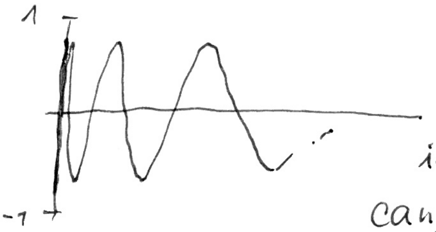
Def: (X,τ) is path-connected if any two points can be connected by continuous path:

$\forall x,y \in X \ \exists \gamma : \langle 0,1\rangle \to (X,\tau) : \gamma(0)=x \ \& \ \gamma(1)=y$

compact interval !!

- counter-example:

$X = \{(x>0,y)\in\mathbb{R}^2 : y = \sin\frac{1}{x}\} \cup [\{0\}\otimes\langle -1,1\rangle]$



- (topologists sine curve)
- not path connected: $(x, \sin\frac{1}{x})$ is infinitely distant from $(0,1)$ $\Rightarrow$ the path can't be parametrized by t from a compact interval

- path-connected $\Rightarrow$ connected ; $\Leftarrow$ NO

Def: (X,τ) is simply-connected if it is path-connected & every path between fixed two points can be continuously transformed to any other path between the same points (paths are homotopic).

$(\Leftrightarrow$ every closed path can be contracted to a point.

- $\Rightarrow$ X does not have holes

Def: Connected component of (X,τ) is any maximal connected subset.

Example: $SO(3) \subset O(3)$ & $O(3) \setminus SO(3) \subset O(3)$

Def: $M \subset (X,\tau)$ is compact if each open cover of M contains finite subcover.

$M \subset \bigcup_{\alpha\in A} \tau_\alpha$, A countable $\Rightarrow$ $\exists B \subset A$ finite : $M \subset \bigcup_{\alpha\in B} \tau_\alpha$

- it is generalization of a bounded & closed Eucl. space (Heine-Borel: $A \subset \mathbb{R}^n$ is compact if it is closed & bounded)
- $M = X \rightarrow (X, \tau)$ is compact space
- continuous image (by a cont. map) of a compact set is compact

- equivalent top. spaces ≡ <u>homeomorphic</u>

<u>Def:</u> <u>Homeomorphism</u> is a mapping $\phi : (X_1, \tau_1) \rightarrow (X_2, \tau_2)$ satisfying

1, ϕ is bijective

2, ϕ & ϕ^{-1} are both continuous

- homeomorphic spaces can be smoothly deformed one to another
- h.s. have equal „topological invariants" (number of holes etc.)

Differentiable manifolds:

<u>Def:</u> Topological manifold M^m of dim. m is topol. space which
1, is Hausdorff (separable)
2, has countable basis
3, is locally homeomorphic to $\mathbb{R}^m$:
$\forall x \in M^m \; \exists U(x) \sim A$ with $A \subset \mathbb{R}^m$ open under standard (metric) topology

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<u>Notes</u> • ad 1, (X, τ) is Hausdorff (T_2) if

$$\forall x, y \in X \quad \exists O_1, O_2 \in \tau : x \in O_1 \;\&\; y \in O_2 \;\&\; O_1 \cap O_2 = \emptyset$$

• ad 2, basis of (X, τ) is a collection $\mathcal{B}$ of open subset such that any $\tau_i \in \tau$ can be written as a countable union of sets from $\mathcal{B}$:

$$\forall \tau_i \in \tau \quad \exists B_\alpha \subset \mathcal{B} : \tau_1 = \bigcup_\alpha B_\alpha$$

• ad 3, std. topology is induced by metric $d(x,y)$:
$B_\epsilon(x) \equiv \{ y \in \mathbb{R}^m / d(x,y) < \epsilon \}$ are open sets

• dimension of M^m is unambiguous

<u>Def:</u> <u>Coordinate chart</u> on M^m is pair (U, ϕ) where $U \subset M^m$ is open set (domain) & $\phi : U \to \mathbb{R}^m$ is homeomorphism from U onto an open subset of $\mathbb{R}^m$ (with std. metric topology).

• if $U = M^m$ then (M^m, ϕ) is a global chart

Def: An atlas on M^m is a collection of charts (U_i, ϕ_i) such that

1, $M = \bigcup_i U_i$

2, $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is C^k $\forall i,j$

- $\phi_j \circ \phi_i^{-1}$ is $\mathbb{R}^m \to \mathbb{R}^m$ → definition of C^k is clear
- an atlas forms differentiable structure on M^m

Def: Differentiable manifold M^m of class C^k and dim. = m is topological manifold dim = m with diff. structure of class C^k.

- smooth manifold – C^∞
- analytical m. – C^ω (analytical functions ≡ absolutely convergent Taylor series on an open neigh. $\sum |a_n| K \Rightarrow \sum a_n A_k$)

LIE GROUPS

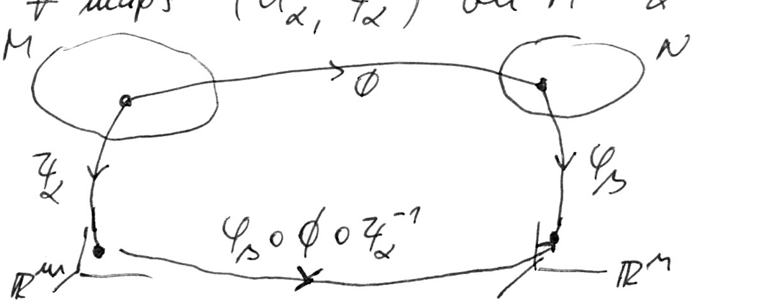
Def: Topological group G is top. space with algebraic structure of a group such that

$\mu : G \times G \to G \qquad (a,b) \mapsto ab$

$\iota : G \to G \qquad a \mapsto a^{-1}$

are continuous mappings.

Def: Mapping $\phi : M^m \to N^n$ between two dif. manifolds is smooth if it is continuous & $\varphi_\beta \circ \phi \circ \psi_\alpha^{-1}$ is C^∞ mapping $\mathbb{R}^m \to \mathbb{R}^n$ $\forall$ maps (U_α, ψ_α) on M^m & (V_β, φ_β) on N^n:



Def: <u>Real Lie group</u> is smooth manifold with algebraic structure of a group such that μ and ι are smooth (C^∞) mappings.

<u>Special cases</u>: • matrix group – group of invertible matrices over the field K

• <u>linear group</u> – group isomorphic to some matrix group (admitting a faithfull finite-dim. representation over K)

<u>GLOBAL PROPERTIES OF LIE GROUPS</u> – examples

1, <u>Euklid group</u> $E(2) = ISO(2)$

$$\hat{x} = R(\alpha)x + a \qquad R(\alpha) = \begin{pmatrix} c\alpha & -s\alpha \\ +s\alpha & c\alpha \end{pmatrix}$$

• 3-dim : α, $a = (a_1, a_2)$

• group operation $\mu : (\alpha, a; \beta, b) \mapsto (\gamma, c)$

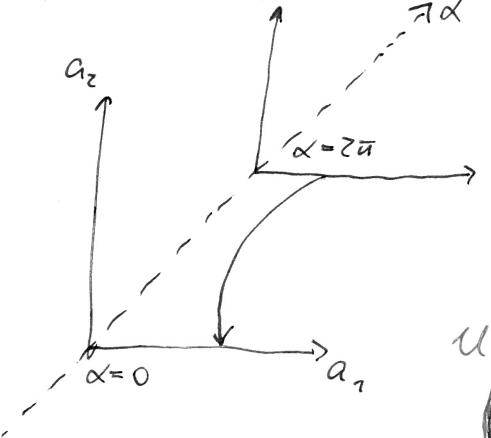
$$\hat{x} = R(\beta)[R(\alpha)x + a] + b \Rightarrow \begin{array}{l} \gamma = \alpha + \beta \\ c = R(\beta)a + b \end{array}$$

• inversion $\iota : (\alpha, a) \mapsto (\gamma, c)$

$$x = R^{-1}(\alpha)x - R^{-1}(\alpha)a \Rightarrow \begin{array}{l} \gamma = -\alpha \\ c = -R(-\alpha)a \end{array}$$

• $\mu, \iota \in C^\infty \Rightarrow$ it is Lie group

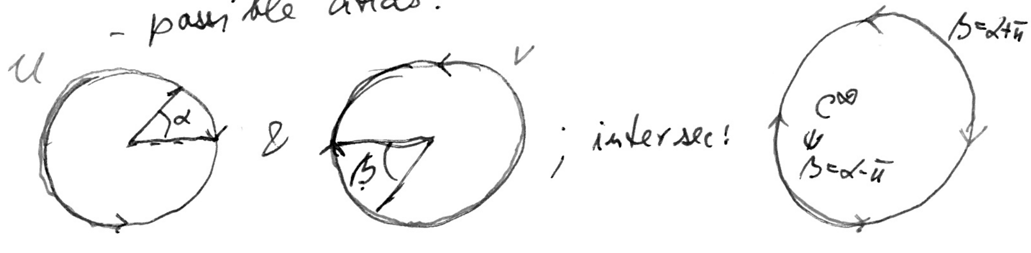
• the manifold is 3D cylinder embedded in $\mathbb{R}^4$:



– the planes $\alpha = 0$ & $\alpha = 2\pi$ are identified

– $\nexists$ global map (it would not be smooth at 2π)

– possible atlas:

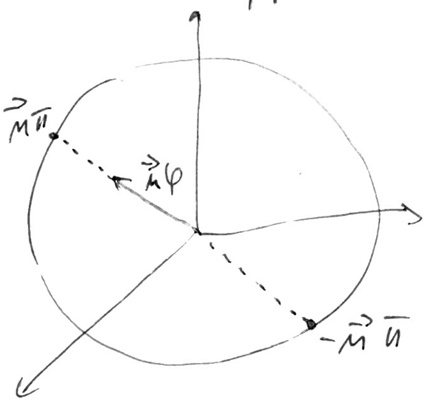


2, SO(2) ... manifold is S^1 - circle of unit diameter (13)

- compact, 1-dim
- multiple-connected (infinitely-fold)
  - two points can be connected by a path which passes n-times around the whole S^1

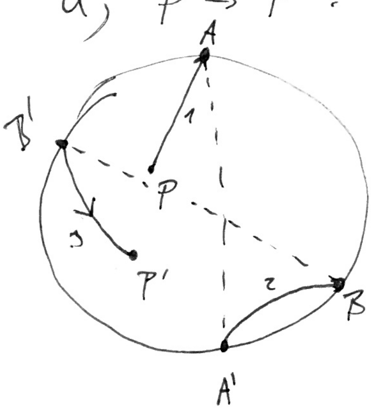
  $\Rightarrow$ paths with different n cannot be deformed to each other $\Rightarrow$ infinitely many classes of paths numbered by n
- SO(2) ~ matrix group $e^{i\varphi}$, $\varphi \in [0, 2\pi)$

3, SO(3) ... 3D ball with diameter of π with identified opposite points on the surface

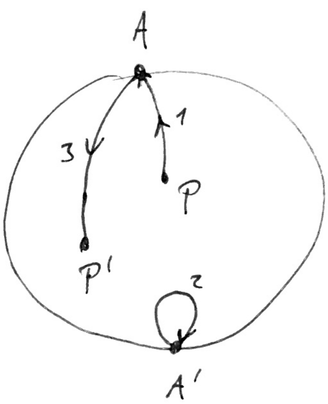


- $\vec{n}\pi = -\vec{n}\pi$
- each point of the ball corresponds to rotation by φ around $\vec{n}$
- compact
- two-fold connected (between each two points there $\exists$ two classes of paths that cannot be deformed between themselves)

a, $P \to P'$:



$B \to A'$
$\longrightarrow$
$B' \to A$



(2) can be contracted to a point $\Rightarrow$ "no jump

$\to$ paths with an even number of jumps ~ paths with no jumps ($P \to P$ can be contracted to a point)

b, paths with odd #. of jumps is another class, $P \to P$ can't be contracted

4, $SU(2)$ – unitary matrices 2×2, $\det u = 1$

$$u = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \quad |a|^2 + |b|^2 = 1 \qquad a, b \in \mathbb{C} \Rightarrow \text{3-dim manifold}$$

$$\Rightarrow u = \mathbb{1}\cos\left(\frac{\omega}{2}\right) - i\,(\vec{\sigma}\cdot\vec{n})\sin\left(\frac{\omega}{2}\right) \qquad [\sigma_i, \sigma_j] = 2i\,\varepsilon_{ijk}\,\sigma_k$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\vec{n}$ … unit vector $\Rightarrow (n_1, n_2, n_3) \leftrightarrow (\vartheta, \varphi)$

- the manifold is 3D ball with diameter of 2π, whole surface corresponds to a single element:
$$u(2\pi, \vartheta, \varphi) = \mathbb{1}_2$$

$\to$ compact, simply-connected manifold (compared to $SO(3)$, jumps are replaced by continuous trajectory on the surface)

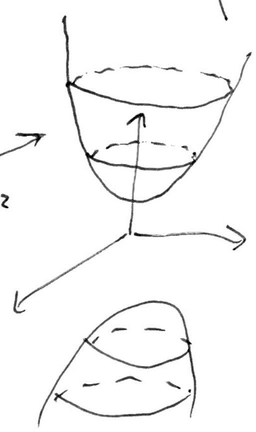
- $su(2) \sim so(3) \Rightarrow SU(2)$ & $SO(3)$ closely connected (but obviously not isomorphic)
  - $\exists\,\phi: SU(2) \to SO(3)$ surjective (2-to-1)
  - $SU(2)$ is universal covering group of $SO(3)$ (resp. $so(3)$)

5, $SL(2,\mathbb{R})$

- regular $\mathbb{R}$ matrices 2×2, $\det = 1$

$$M = \begin{pmatrix} x_1 & x_2 \\ x_3 & \frac{1+x_2+x_3}{x_1} \end{pmatrix} \quad \text{for } x_1 = 0 \quad \text{or} \quad M = SO = \begin{pmatrix} z+y & x \\ x & z-y \end{pmatrix}\begin{pmatrix} c\varphi & s\varphi \\ -s\varphi & c\varphi \end{pmatrix}$$

- $S = (MM^T)^{1/2}$ … sym, $\det = 1$
  $(z^2 - x^2 - y^2 = 1)$ $\quad z^2 = 1 + x^2 + y^2$
- $O = S^{-1}M$ … og., $\det = 1$



$\otimes$

$= H^2 \otimes S^1$

- non-compact, not connected