Derived homomorphism of LA

NB: • homomorphism between LG = smooth algebraic homomorphism,
• isomorphism between LG = diffeomorphic homomorphism

Def: Homomorphism between L. algebras is linear map $\Psi : \mathcal{G} \to \mathcal{G}'$ such that ($\mathcal{G}$ & $\mathcal{G}'$ are over F)

1, $\Psi(\alpha X + \beta Y) = \alpha \Psi(X) + \beta \Psi(Y)$ $\quad \forall X, Y \in \mathcal{G}$

2, $\Psi([X, Y]) = [\Psi(X), \Psi(Y)]$ $\quad \forall \alpha, \beta \in F$

Theorem: Let $\phi : G \to H$ be homomorphism between LG and $g(t) = \exp(tX)$, $X \in \mathcal{G}$ is one-param. subgroup in G. Then

1, $\phi(g(t)) = h(t) \subset H$ is one-param. subgroup of H given by
$$h(t) = \exp(tY) \quad Y = \phi_* X \in T_e H$$

2, $\phi_* : T_e G \to T_e H$ is <u>derived homomorphism</u> between $\mathcal{G}$ & $\mathcal{H}$.

• G, H homomorphic $\Rightarrow$ $\mathcal{G}, \mathcal{H}$ homomorphic

• corresponding commutative diagram

$$\begin{tikzcd} G \arrow[r, "\phi"] & H \\ \mathcal{G} \arrow[u, "\exp"] \arrow[r, "\phi_*"] & \mathcal{H} \arrow[u, "\exp"'] \end{tikzcd} \quad \Leftrightarrow \quad \boxed{\phi \circ \exp = \exp \circ \phi_*}$$

Proof:

1, ϕ homomorf $\Rightarrow$ $h(t+s) = \phi(g(t+s)) = /\, g$ 1-param subgr. $/$

$= \phi(g(t))\,\phi(g(s)) = h(t)\,h(s)$

$\Rightarrow \exists\, Y \in T_e H : \; h = \exp(tY)$

$$Y(f) \equiv \frac{d}{dt}\left(f(h(t))\right)\Big/_{t=0} = \frac{d}{dt} f(\phi(g(t)))\Big/_{t=0} \equiv (\phi_* X)(f)$$

$\Rightarrow Y = \phi_* X$

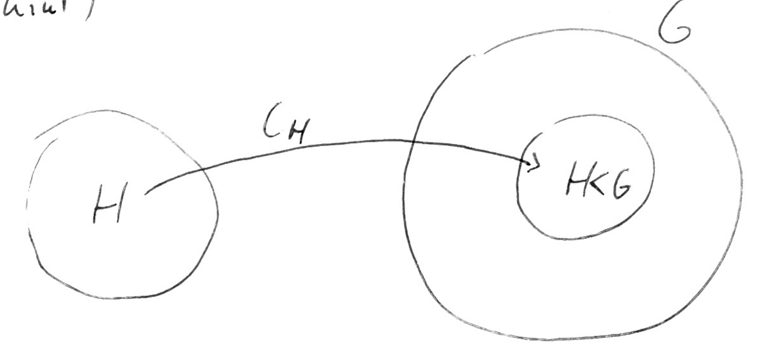
2, show that ϕ_* acting on left-invar. field is linear and preserves commutator (exc.)

hint: • $c(\sqrt{t}) = g(\sqrt{t})\, g'(\sqrt{t})\, g(\sqrt{t})^{-1}\, g'(\sqrt{t})^{-1}$ is one-param. subgroup generated by $[X, X'] \in \mathcal{G}$ (Pontryagin)

• or use coord. repre

Theorem: Let $H < G$ be a Lie subgroup. Then
$$\mathcal{H} = \{ X \in \mathcal{G} \mid \exp_G(tX) \in H \;\; \forall t \in \mathbb{R} \}$$
is LA of H and subalgebra of $\mathcal{G}$.

Proof: (hint)



$\iota_H : H \hookrightarrow G$ is smooth immersion

$\iota_H(h) = h \in G$

$\Rightarrow (\iota_H)_*$ is derived homomorphism $\mathcal{H} \to \mathcal{G}$

$\Rightarrow$ $\mathcal{H}$ & $\mathcal{G}$ has "the same" commutator

RELATIONS BETWEEN L. groups & L. algebras

- what are L. algebras of isomorphic groups?
- what are L. groups with isomorphic algebras?

Theorem: Let $\phi : G \to G$ be isomorphism between LG. Then the derived homomorphism $\phi_* : \mathcal{G} \to \mathcal{G}'$ is isomorphism.

Proof: 1, ϕ_* is injective:

$\phi_*(X) = \phi_*(Y)$ for $X, Y \in \mathcal{G}$:

$\to \exp(t\phi_*(X)) \equiv \phi(\exp(tX)) = \phi(\exp(tY)) \equiv \exp(t\phi_*(Y))$

$\Rightarrow$ /ϕ injective/ $\to \exp(tX) = \exp(tY) \quad \forall t$

$\to X = Y$ (there is 1-to-1 corresp. between X & $\gamma^X(t)$)

2, ϕ_* is surjective as every $\gamma'(t) \subset G'$ has a pre-image in G (ϕ surjective & homomorphic)

$\Rightarrow$ every $X' \in \mathcal{G}'$ has pre-image in $\mathcal{G}$. $\boxtimes$

Def: Subgroup $H < G$ of a Lie group G is discrete if it is finite or countable & $\exists\, \mathcal{U}(e) \subset G$ which does not contain any element of H other than e.

Theorem: Let $\phi : G \to G'$ be smooth surjective homomorphism between LG & let $\mathrm{Ker}\,\phi < G$ is a discrete subgroup. Then $\phi_* : \mathcal{G} \to \mathcal{G}'$ is isomorphism.

Proof: (hint)

- $\mathrm{Ker}\,\phi$ discrete $\to \exists\, \mathcal{U}(e_G) : (\mathrm{Ker}\,\phi) \cap \mathcal{U}(e_G) = \{e_G\}$

$\Rightarrow \phi : \mathcal{U}(e_G) \to \mathcal{U}(e_{G'})$ is isomorphism ($\to \dim G = \dim G'$)

$\Rightarrow \mathcal{G} \sim \mathcal{G}'$

Universal covering group

NB: • center $Z(G) = \{h \in G \mid hg = gh \ \forall g\}$ is abelian normal subgroup of G

• $H < G$ is central if $H \subset Z(G)$

Theorem: Let G be a connected LG. Then there exists a simply connected LG $\bar{G}$ (unique up to isomorphism) such that:

a, G is isomorphic to a (Lie) factor group $\bar{G}/K$, where K is some discrete central subgroup of $\bar{G}$

b, if G is simply connected then $G \sim \bar{G}$

c, $\mathcal{G} \sim \bar{\mathcal{G}}$ (real LA of $G, \bar{G}$)

• $\Rightarrow$ for every LA $\mathcal{G}$ $\exists!$ simply connected "universal covering group"

• K is kernel of some homomorphism, in particular
$$\pi: \bar{G} \to \bar{G}/K \qquad \bar{g} \mapsto \bar{g}K$$

Note: Covering space of a top. space (X, τ) is the topol. space (C, ρ) together with continuous surjective map
$$p: (C, \rho) \xrightarrow{\text{onto}} (X, \tau) \quad \text{such that}$$
$$\forall x \in X \quad \exists\, \mathcal{U}(x) \in \tau : p^{-1}(\mathcal{U}(x)) = \bigcup_\alpha V_\alpha \quad \text{with}$$
$V_\alpha \subset (C, \rho)$, $V_\alpha \cap V_{\alpha'} = \emptyset$ & V_α are homeomorphic to $\mathcal{U}(x)$ through the map p.

$$\begin{tikzcd}
 & & & \boxed{\bar{G}} \arrow[llld, "\pi_1"'] \arrow[lld, "\pi_2"] \arrow[rrd, "\pi_{max}"] \arrow[dd, shift left, "lin"] & & \\
\boxed{G/K_1} \arrow[rrrd, "lin"'] & \boxed{G/K_2} \arrow[rrd, "lin"] & \cdots & & \cdots & \boxed{G/K_{max}} \arrow[llld, "lin"] \\
 & & & \boxed{\mathfrak{g}} \arrow[uu, shift left, "exp"] & &
\end{tikzcd}$$

- K_{max}: max. discrete central subgroup of $\bar{G}$
- finding K_{max} for lin. Lie groups is simple: lin. Lie group has faithful finite-dim repre; if it is irreducible then $A \in K_{max} \Rightarrow A = \lambda \mathbb{1}$ (Schur)

Example: 1, $SU(2)$ simply connected $\Rightarrow$ it is the universal cover of $SU(2) \sim SO(3)$

$\Rightarrow \overline{SO(3)} = SU(2)$

- $SO(3) \sim SU(2)/\{-\mathbb{1}, \mathbb{1}\}$ (exercice)

& $\{\mathbb{1}, -\mathbb{1}\}$ is the only nontrivial discrete central subgroup of $SU(2)$

$\Rightarrow SU(2)$ & $SO(3)$ are the only two L. groups with the $su(2)$ algebra; $SU(2) \to SO(3)$ is double covering

2, $\overline{SO(2)} = (\mathbb{R}, +)$

$\pi : (\mathbb{R}, +) \to SO(2) \quad \varphi \mapsto e^{i\varphi}$ is infinite-fold cover $(e^{i\varphi} = e^{i(\varphi + 2\pi k)})$