Some structure theory of LA

NB: • subalgebra = vec. subspace closed with resp. to $[\cdot,\cdot]$

• $\mathcal{H} \subsetneq \mathcal{G} \Rightarrow \dim \mathcal{H} < \dim \mathcal{G}$ (at least one generator must be missing

Def: Subalgebra $\mathcal{H} \subset \mathcal{G}$ is invariant (ideal) if
$[X,Y] \in \mathcal{H} \quad \forall X \in \mathcal{H} \quad \forall Y \in \mathcal{G}$

Def: $\mathcal{G}$ is semisimple if it does not contain proper abelian invariant subalgebra.

NB: semisimple LG does not contain proper abelian normal subgroup $(gHg^{-1} = H)$ -> semisimple LA

Def: $\mathcal{G}$ is simple if it is non-abelian and does not contain any proper invar. subalgebra

$\Updownarrow$

NB: simple LG does not contain any proper normal subgroup

(• simple & semi-simple LG important in particle physics
• interesting theoretically – $\exists$ complete classification (cf. Cartan subalgebras, Dynkin diagrams))

Theorem: For a finite-dim. LG G, $H \triangleleft G \Rightarrow \mathcal{H} \subset \mathcal{G}$ is invariant subalgebra.

Proof: • we already know that $\mathcal{H} \subset \mathcal{G}$ is subalgebra (cf. derived homomorphism)

• $H \triangleleft G \Rightarrow C(\sqrt{t}) = h(\sqrt{t})\, g(\sqrt{t})\, h(\sqrt{t})^{-1} g(\sqrt{t})^{-1} \subset H$ 1-param. subgroup & $C(\sqrt{t}) = \exp([A,B]t)$ for $h(t) = e^{tA}$, $g(t) = e^{tB}$
$\Rightarrow [A,B] \in \mathcal{H}$, $A \in \mathcal{H}$, $B \in \mathcal{G}$ arb.

Intermezzo: adjoint representation of LA

Theorem: Let G be real (or complex) LA, $\dim G = n$, & let $e_1, \dots, e_n$ be a basis of G. Define for $X \in G$ $n \times n$ matrix $\underline{ad(X)}$ by

$$[X, e_j] = \sum_{k=1}^{n} (ad(X))^k_j \, e_k \qquad j = 1, \dots, n$$

Then the matrices $ad(X)$ form n-dimensional adjoint representation of G.

NB: • repre of G = homomorphism $G \to$ matrix algebra preserving $[\cdot,\cdot]$

• $(ad(e_i))^k_j = c^k_{ij} \Rightarrow$ we have already seen ...

Proof: • $ad(X)$ is well defined: $X \in G \Rightarrow [X, e_j] \in G \Rightarrow [X, e_j] = \sum q^k e_k$

• $[\cdot,\cdot]$ linear $\Rightarrow ad(\alpha X + \beta Y) = \alpha\, ad(X) + \beta\, ad(Y)$

• $ad([X,Y]) = [ad(X), ad(Y)]$ follows from Jacobi (see (6))

$$[[X,Y], e_j] = (ad([X,Y]))^k_j \, e_k$$

$\parallel$ Jacobi

$$-[[Y, e_j], X] + [[X, e_j], Y] = -(ad(Y))^k_j [e_k, X] + (ad(X))^k_j [e_k, Y]$$

$$= (ad(Y))^k_j (ad(X))^l_k \, e_l - (ad(X))^k_j (ad(Y))^l_k \, e_l$$

$$= [ad(X), ad(Y)]^l_j \, e_l \qquad \boxtimes$$

- $ad(X)$ <u>is an action of $\mathcal{G}$ on itself</u>:

$$ad: \mathcal{G} \times \mathcal{G} \to \mathcal{G} \qquad ad(X)Y = [X,Y]$$

$$Y = q^j e_j \Rightarrow [X,Y] = q^j [X, e_j] = q^j (ad(X))^k_j \, e_k$$

$$= \tilde{q}^k e_k \Rightarrow \tilde{q} = ad(X) q$$

- $ad(X)$ is lin. operator (matrix) $\Rightarrow$ under a basis transformation $S: e_i \mapsto e_i'$ it transforms as

$$S: ad(X) \mapsto S^{-1} ad(X) S$$

<u>Def</u>: <u>Killing - Cartan form</u> is symmetric bilinear map $B: \mathcal{G} \times \mathcal{G} \to \mathbb{R}/\mathbb{C}$

$$B(X,Y) \equiv Tr[ad(X) ad(Y)] \qquad \forall X,Y \in \mathcal{G}$$

- for a real $\mathcal{G}$, matrix elements $ad(X) \in \mathbb{R}$
  $\to B(X,Y) \in \mathbb{R}$
  ! need not to be true for arb. repre - cf. Pauli matrices
- $\mathcal{G}$ complex $\Rightarrow B(X,Y) \in \mathbb{C}$
- $B(X,Y)$ invariant under all transf. $S \in Aut(\mathcal{G})$
  $\Rightarrow$ is indep. of the basis choice
  $$Tr(S^{-1} ad(X) S S^{-1} ad(Y) S) = Tr(ad(X) ad(Y))$$

<u>Def</u>: <u>Killing - Cartan metric</u> on $\mathcal{G}$ is defined as

$$g_{ij} \equiv B(e_i, e_j) = c^k_{il} \, c^l_{jk} = g_{ji}$$

- transforms as a 2nd-rank tensor
- well defined due to basis independence of $B(\cdot,\cdot)$

Examples: 1, $su(2)$

- $c_{ij}^{k} = \varepsilon_{ijk} \Rightarrow ad(e_i)_j^k = \varepsilon_{ijk}$ & $\varepsilon_{ijs}\varepsilon_{kls} = \delta_{ik}\delta_{lj} - \delta_{il}\delta_{jk}$

$\Rightarrow Tr[ad(e_i)\,ad(e_j)] = ad(e_i)_k^l \, ad(e_j)_l^k$

$= \varepsilon_{ikl}\varepsilon_{jlk} = -\varepsilon_{ikl}\varepsilon_{jkl} = -\delta_{ij}\delta_{kk} + \delta_{ik}\delta_{jk} = -2\delta_{ij}$

$\Rightarrow B(e_i, e_j) = \boxed{g_{ij} = -2\delta_{ij}}$

- g_{ij} is non-degenerate : $\det g = -8 \Rightarrow \nexists X : g(X,Y) = 0 \; \forall Y$

$\Rightarrow g_{ij}$ defines inner product on $su(2)$

2, $gl(n,\mathbb{R})$ (exerc.)

- Weyl basis $\to B(X,Y) = 2n\,Tr(XY) - 2Tr(X)\,Tr(Y)$
- g_{ij} degenerate : $B(\mathbb{1}, Y) = 0 \;\; \forall Y$

3, $sl(n,\mathbb{R})$: $Tr(X) = 0 \Rightarrow B(X,Y) = 2n\,Tr(XY)$

Theorem: (Properties of K-C form on $\mathcal{G}$)

1, $B(Y,X) = B(X,Y)$

2, $B(\alpha X, \beta Y) = \alpha\beta B(X,Y)$

3, $B(X, Y+Z) = B(X,Y) + B(X,Z)$

4, $\varphi \in Aut(\mathcal{G}) \Rightarrow B(\varphi(X), \varphi(Y)) = B(X,Y)$

5, $B([X,Y], Z) = B(X, [Y,Z])$

6, $\mathcal{G}'$ ideal of $\mathcal{G}$ & $B_{\mathcal{G}'}$ is K-C form on $\mathcal{G}'$

$\Rightarrow B_{\mathcal{G}}(X,Y) = B_{\mathcal{G}'}(X,Y) \quad \forall X,Y \in \mathcal{G}'$

Theorem (Cartan 1st criterion)

LA $\mathcal{G}$ is solvable $\Leftrightarrow B(X,Y)=0 \quad \forall X,Y \in \mathcal{G}^{(1)}=[\mathcal{G},\mathcal{G}]$

NB: 1, $\mathcal{G}$ solvable if $\exists n>0: D^n\mathcal{G}=0$, where $D^k\mathcal{G}$ is defined through

- $D^0\mathcal{G}=\mathcal{G}$ & $D^{k+1}\mathcal{G}=[D^k\mathcal{G},D^k\mathcal{G}]$
- $[\mathcal{G},\mathcal{G}']=\{[A,B]\,/\,A\in\mathcal{G}\ \&\ B\in\mathcal{G}'\}$

2, $\mathcal{G}$ semi-simple $\Leftrightarrow$ does not contain non-trivial solvable ideal

Theorem (Cartan 2nd criterion)

LA $\mathcal{G}$ semi-simple $\Leftrightarrow B(X,Y)$ non-degenerate

$(\Leftrightarrow \det g_{ij}\neq 0)$

Theorem: LG G is compact $\Leftrightarrow B(X,Y)$ on corresp. LA $\mathcal{G}$ is negative definite.

- cf. $su(2)\sim so(3)$ & $g_{ij}=-2\delta_{ij}$

Corollary: Every compact LG is semi-simple.

-> $Sl(2,\mathbb{R})$ - Gilmore

-> 54 ...

-> repre of LA/LG

REPRESENTATIONS OF LIE ALGEBRAS

Def: Repre of of a LA $\mathcal{G}$ on lin. vect. space V is homomorphism

$$\rho : \mathcal{G} \to End(V) \qquad \text{(ie, incl. } Ker\, V \neq \emptyset)$$

Def: d-dim matrix repre of LA over field F is association of every $X \in \mathcal{G}$ with a $d \times d$ matrix $D(X)$ satisfying

$$\text{i)}\ D(\alpha X + \beta Y) = \alpha D(X) + \beta D(Y)$$
$$\text{ii)}\ D([X,Y]) = [D(X), D(Y)] \qquad \forall X, Y \in \mathcal{G},\ \forall \alpha, \beta \in F$$

- it is sufficient to find matrices of the generators/ basis of $\mathcal{G}$
- null (trivial) repre: $D(X) = 0 \quad \forall X$
- reducibility & complete reducibility; equivalence of repre defined identically as for groups
- reducibility only needs to be checked for generators

Theorem: (Schur lemmas)

1) Let D & D' be two IRREPs of $\mathcal{G}$ of dims. d & d' & let A be $d \times d'$ matrix such that

$$D(X) A = A D(X) \quad \forall X \in \mathcal{G}.$$

Then either $A = 0$ or $d = d'$ & $\det A \neq 0$.

2) Let D be a d-dim IRREP of $\mathcal{G}$ & B is a $d \times d$ matrix such that

$$D(X) B = B D(X) \qquad \forall X \in \mathcal{G}.$$

Then $D = \lambda \mathbb{1}_d$

$\Rightarrow$ IRREPs of Abelian G are 1-dim.

<u>Def</u>: <u>Analytical repre</u> of a LG G is a repre such that the matrix elements of $D(g(x_1, \dots, x_n))$ are analytical functions of loc. coordinates on $\mathcal{U}(e)$.

$\Rightarrow$ mat. elements can be expanded in Taylor series on $\mathcal{U}(e)$ $\rightarrow$ left transl. $\rightarrow$ it holds on the whole group

<u>Theorem</u>: Let D_G is d-dim anal. repre of G with a LA $\mathcal{G}$. Then

1, matrices $D_{\mathcal{G}}$ def. for every $X \in \mathcal{G}$ as

$$D_{\mathcal{G}}(X) = \frac{d}{dt} D_G(\exp tX)\Big|_{t=0} \qquad (*)$$

form d-dim repre of $\mathcal{G}$ & for $\forall X \in \mathcal{G}$ $\forall t \in \mathbb{R}$,

$$\exp(t D_{\mathcal{G}}(X)) = D_G(\exp(tX)) \qquad (**)$$

(need not cover whole G but $\exists$ $\forall t \in \mathbb{R}$)

2, Let D_G & D_G' be two d-dim anal. repre of G & $D_{\mathcal{G}}, D_{\mathcal{G}}'$ corresp. repre of $\mathcal{G}$ def. throug $(*)$.

Then $D_G \sim D_G' \Rightarrow D_{\mathcal{G}} \sim D_{\mathcal{G}}'$ ($\Leftarrow$ for connected G)

3, D_G reducible $\Rightarrow$ $D_{\mathcal{G}}$ reducible ($\Leftarrow$ for -"- G)

4, D_G fully red. $\Rightarrow$ $D_{\mathcal{G}}$ fully reducible (-"-)

5, Let G be connected. Then D_G IR $\Leftrightarrow$ $D_{\mathcal{G}}$ IR.

6, D_G unitary $\Rightarrow$ $D_{\mathcal{G}}$ anti-hermitian ($\Leftarrow$ for connected G)

Proof:

1, (**) is derived homomorphism $\rho_* : \mathcal{G} \to End(V)$
generated by $\rho : G \to Aut(V)$

(*) follows from (**) via derivative due to analyticity

2, • $\exp(S^{-1} g S) = S^{-1} \exp(g) S$

$\boxed{\Rightarrow}$ $D_G \sim D_G' \Rightarrow D_{\mathcal{G}}'(X) = \frac{d}{dt}\left(S^{-1} D_G(\exp tX) S\right)\Big/_{t=0}$

$\overset{(**)}{=} \frac{d}{dt} S^{-1} \exp(t D_{\mathcal{G}}(X)) S = \frac{d}{dt} \exp(t S^{-1} D_{\mathcal{G}}(X) S)$

$= S^{-1} D_{\mathcal{G}}(X) S$

$\boxed{\Leftarrow}$ G connected $\Rightarrow g = \prod_i \exp(t_i X_i) \quad \forall g \in G$

$D_{\mathcal{G}}' \sim D_{\mathcal{G}} \overset{(**)}{\Rightarrow} D_G'(\exp(tX)) = \exp(t S^{-1} D_{\mathcal{G}}(X) S)$ $\boxtimes$

3&4, • $\boxed{\Rightarrow}$ directly from (*)
• $\boxed{\Leftarrow}$ block structure reproduced at all powers of a matrix ~~&~~ (**) & $g = \prod_i \exp(t_i X_i)$

5, directly from J

6, directly from (**)

$\boxtimes$

Note: • not every $D_{\mathcal{G}}$ gives through exp valid repre of G

• only if D_G is analytical, then it can be recovered from the corresp. repre of $\mathcal{G}$

$\Rightarrow$ not all repre of $\mathcal{G}$ can be obtained by derivative of some repre of G; unles G is simply connected (universal covering gr.)

Example 1) $SO(2)$, $so(2)$

- $so(2)$ 1-dim, $e_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow \exp(te_1) = \begin{pmatrix} ct & st \\ -st & ct \end{pmatrix}$ (+)
- str. constants: $[e_1, e_1] = 0$

$\Rightarrow D_{\mathcal{G}}(e_1) = p$ is 1-dim repre $\mathcal{G}$ for arb. $p \in \mathbb{C}$

$\Rightarrow \exp(t D_{\mathcal{G}}(e_1)) = \exp(tp)$ (++)

- (+) $\Rightarrow \exp((t+2\pi)e_1) = \exp(te_1)$

(++) $\Rightarrow \exp((t+2\pi)p) = \exp(2\pi p)\exp(tp)$

$\Rightarrow \exp(tp)$ is repre of $SO(2)$ only for $p = ik$, $k \in \mathbb{Z}$

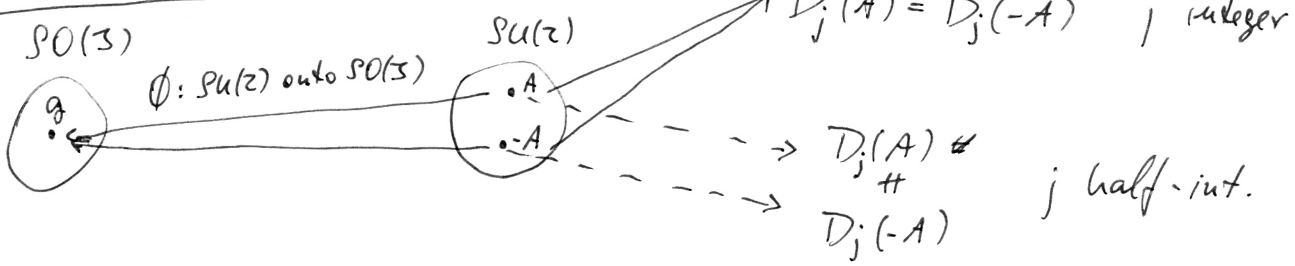
2) $SO(3)$ & $so(3) \sim su(2)$

- $\exists$ 2-dim repre $so(3)$ given by Pauli matrices

$$D_{\mathcal{G}}(e_1) = \frac{1}{2}\sigma_1 \,;\; D_{\mathcal{G}}(e_2) = \frac{1}{2}\sigma_2 \,;\; D_{\mathcal{G}}(e_3) = \frac{1}{2}\sigma_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$\Rightarrow \exp(t D_{\mathcal{G}}(e_3)) = \begin{pmatrix} \exp(\frac{1}{2}ti) & 0 \\ 0 & \exp(\frac{1}{2}ti) \end{pmatrix} \overset{t = t+2\pi}{\Rightarrow} -\begin{pmatrix} \cdot & 0 \\ 0 & \cdot \end{pmatrix} \quad ↯$$

$\Rightarrow$ representation is multi-valued



- let $\bar{G}$ be univ. cover. group of $\mathcal{G}$ $\Rightarrow$ $D_{\bar{G}}$ gives repre of $G \sim \bar{G}/K$ throug the homomorphism $\phi: \bar{G} \to G$ with ~~a~~ discrete kernel K,

$D_G(\phi(g)) = D_{\bar{G}}(g)$, if for $\forall k_i \in K$ $D_{\bar{G}}(k_i g) = D_{\bar{G}}(g)$ $\forall g \in \bar{G}$

COMPLEXIFICATION OF A LA

- how to make $\mathbb{C}$ LA out of $\mathbb{R}$ LA?
- motivation: construction of IRREPS
  cf. ladder operators $L_\pm = L_1 \pm i L_2$

CASE 1: generators of $\mathbb{R}$ LA are LN also over $\mathbb{C}$:

$\sum \lambda_i e_i = 0 \Rightarrow \lambda_i = 0 \quad \forall i, \lambda_i \in \mathbb{C}$

$\Rightarrow$ straightforward

Example: • $su(2)$ as $\mathbb{R}$ LA of anti-herm. matrices

has basis $e_i = \pm \frac{i}{2}\sigma_i$

$$e_1 = \frac{i}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad e_2 = -\frac{i}{2}\sigma_2 = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix} \qquad e_3 = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Rightarrow \big/ [\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k \big/ \Leftrightarrow [e_i, e_j] = \varepsilon_{ijk} e_k$$

$\Rightarrow e_i$ LN also over $\mathbb{C}$, but $su(2)_{\mathbb{C}}$ is no longer LA of anti-herm. matrices ($i \times$ anti-herm $\neq$ anti-herm.)

CASE 2: generators of $\mathbb{R}$ LA lin. dependent over $\mathbb{C}$:

Example: • $sl(2,\mathbb{C})$ as $\mathbb{R}$ LA is 6-dim (traceless mat.):

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$e_4 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = ie_1 \qquad e_5 = ie_2 \qquad e_6 = ie_3$$

$\Rightarrow sl(2,\mathbb{C})^{\mathbb{C}} \sim su(2)_{\mathbb{C}}$ is only 3-dim

- however, for $\mathbb{R}$ LA $sl(2,\mathbb{C}) \sim o(1,3)$ & direct complexification $o(1,3)_{\mathbb{C}}$ is 6-dim

$\Rightarrow$ there must $\exists$ 6-dim complex LA, which is complexification of $sl(2,\mathbb{C})$

CONSTRUCTION:

- let $e_1, \dots, e_n$ be basis of a $\mathbb{R}$ CAG, which can be lin. dep. over $\mathbb{C}$
- construct $2n$-dim $\mathbb{R}$ vec. space with elements

$$(X, Y) \qquad X, Y \in G \qquad \text{with operations}$$

1, $\alpha(X,Y) = (\alpha X, \alpha Y) \qquad \alpha \in \mathbb{R}$

2, $(X,Y) + (X',Y') = (X+X', Y+Y')$

- transform to $\mathbb{C}$ vect. space $G_\mathbb{C}$ with scalar multiplication

$$(\alpha + i\beta)(X,Y) = (\alpha X - \beta Y, \alpha Y + \beta X) \qquad \alpha, \beta \in \mathbb{R}$$

Ex: 1, $z_1(z_2(X,Y)) = (z_1 z_2)(X,Y)$

2, $(z_1 + z_2)(X,Y) = z_1(X,Y) + z_2(X,Y)$

3, $z[(X,Y) + (X',Y')] = z(X,Y) + z(X',Y')$

$\Rightarrow$ $G_\mathbb{C}$ is lin vect. space

- $\dim G_\mathbb{C} = n$: basis is $(e_1, 0), \dots, (e_n, 0)$

$$(0, e_j) = i(e_j, 0)$$

- commutator on $G_\mathbb{C}$:

$$[(X,Y),(X',Y')] = ([X,X'] - [Y,Y'], [X,Y'] + [Y,X'])$$

$\Rightarrow$ $G_\mathbb{C}$ forms $\mathbb{C}$ LA

Exercise: • verify anti-sym. & Jacobi identity

- structure constants on $G_\mathbb{C}$:

$$[(e_i,0),(e_j,0)] = ([e_i,e_j],0) = \sum_k c_{ij}^k (e_k, 0)$$

$\Rightarrow$ $G_\mathbb{C}$ has the same c_{ij}^k as G

- if e_i are CN over $\mathbb{C}$, $\mathcal{G}_\mathbb{C}$ is isomorphic to straightforw. complexification throug the map

$$\phi((X,Y)) = X + iY$$

- c_{ij}^k the same $\Rightarrow$ $ad_\mathcal{G}(e_i) = ad_{\mathcal{G}_\mathbb{C}}((e_i,0))$

$$\Rightarrow B_\mathcal{G}(e_i,e_j) = B_{\mathcal{G}_\mathbb{C}}((e_i,0),(e_j,0))$$

Cartan II. $\Rightarrow$ $\underline{\mathcal{G} \text{ semi-simple} \Leftrightarrow \mathcal{G}_\mathbb{C} \text{ semi-simple}}$

- simple LA - only one implication

$$\mathcal{G}_\mathbb{C} \text{ simple} \Rightarrow \mathcal{G} \text{ simple}$$

<u>Ex:</u> • $so(1,3)$ simple, $so(1,3)_\mathbb{C}$ not simple

- however, if $\mathcal{G}$ simple & $\mathcal{G}_\mathbb{C}$ not simple, then $\mathcal{G}_\mathbb{C}$ is direct sum of two isomorphic LAs:

$$so(1,3)_\mathbb{C} \sim su(2)_\mathbb{C} \oplus su(2)_\mathbb{C}$$ (see 53)

Representations of complexified LA

- if $D_\mathcal{G}(e_i)$ is repre of $\mathcal{G}$, then $D_{\mathcal{G}_\mathbb{C}}((e_i,0)) = D_\mathcal{G}(e_i)$ is repre of $\mathcal{G}_\mathbb{C}$:

$$\rightarrow [D_{\mathcal{G}_\mathbb{C}}((e_i,0)), D_{\mathcal{G}_\mathbb{C}}((e_j,0))] = [D_\mathcal{G}(e_i), D_\mathcal{G}(e_j)] \underset{D_\mathcal{G} \text{ repre}}{=} D([e_i,e_j])$$

$$\rightarrow D_{\mathcal{G}_\mathbb{C}}([(e_i,0),(e_j,0)]) = D_{\mathcal{G}_\mathbb{C}}([e_i,e_j],0) = D_\mathcal{G}([e_i,e_j]) \quad \checkmark$$

- through same reasoning, repre of $\mathcal{G}_\mathbb{C}$ generates repre of $\mathcal{G}$
- obviously $\boxed{D_\mathcal{G} \text{ IRREP} \Leftrightarrow D_{\mathcal{G}_\mathbb{C}} \text{ IRREP}}$

$\Rightarrow$ IRREPs of $\mathcal{G}$ can be constructed via complexification