GROUP ACTION (on a set)

Def: Let G be a group and M a set. It is said that G is acting on M if there exists a mapping

$$\varphi : G \times M \to M \qquad \varphi(g,m) \equiv T(g)m \equiv gm$$

such that $\forall g,h \in G \quad \forall m \in M$

a, $T(g)T(h)m = T(gh)m$

b, $T(e)m = m$

- $T(g)$ is a transformation on M assigned to g
- action is a homomorphism from G to the group of transformations on M (needs not to be injective)

Def: An orbit of an element $m \in M$ under the action of G is the set

$$G.m = \{T(g)m \,/\, g \in G\} \subset M$$

- orbits define equivalence relation on M and partition M to equiv. classes $G.m$

Def: Stabilizer (isotropy) group with respect to $m \in M$ is the subset of G

$$G_m = \{g \in G \,/\, T(g)m = m\} \subset G$$

Lemma: G_m is a subgroup of G

Proof: • $T(e) = \mathbb{1}$ by def. of action $\Rightarrow e \in G_m$

• $g \in G_m \Rightarrow g^{-1} \in G_m$: $m = T(e)m = T(g^{-1}g)m$
$= T(g^{-1})m$

• $g, g' \in G_m \to T(g)T(g')m = m = T(gg')$

$\boxtimes$

Theorem 10

Let G be a finite group acting on a set M.
Then $(\# G\cdot m)\cdot(\# G_m) = \# G \quad \forall m \in M$.

Proof: • $n \in G\cdot m \Rightarrow \exists g \in G : T(g)m = n$

• let $m \in M$ and $g \in G$ be fixed and let $\exists g' \in G$:

$n = T(g)m = T(g')m \Rightarrow T(g^{-1}g')m = m$

$\Rightarrow g^{-1}g' \in G_m$

$\Rightarrow$ for fixed g & m there exist exactly $\# G_m$ elements $g' \in G : g^{-1}g' \in G_m$ (rearrang. th.)

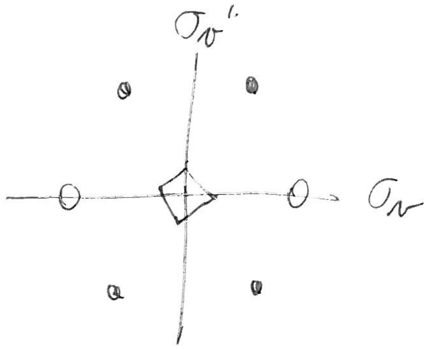
• each $g \in G$ maps m to some element from $G\cdot m$

$\Rightarrow$ it is possible to choose exactly $\#(G\cdot m)$ "non-equivalent" elements of G, each representing the $\# G_m$ elements mapping m to a fixed element of $G\cdot m$

$\Rightarrow \# G = (\# G\cdot m)(\# G_m)$ $\boxtimes$

• In other words, each element of G maps m somewhere to the orbit and ~~to~~ each element of the orbit maps exactly $\# G_m$ elements of G.

Example: C_{2v} on $\mathbb{R}^3$



• $\Rightarrow \#(G\cdot m) = 4, \quad G_m = \{E\}$

○ $\Rightarrow \#(G\cdot m) = 2, \quad G_m = \{E, \sigma_v\}$

◇ $\Rightarrow \#(G\cdot m) = 1, \quad G_m = C_{2v}$

Group action on itself: $(G \times G \to G)$

1, left/right translation

$L_g : G \to G \quad h \mapsto gh$

$R_g : G \to G \quad h \mapsto hg$ $\quad \forall h \in G,\ g \in G$ fixed

- $L_g / R_g : G \to G$ is an isomorphism (Rearr. theorem)
- L_g, R_g are transitive : $G.h = G$, $G_h = \{e\}$ for each $h \in G$

2, conjugation (inner automorphism)

$T(g)h \equiv ghg^{-1} \quad \forall h \in G$

- $G.h = (h)$

  $G_h = \{g \in G \mid gh = hg\}$ is a subgroup of G

  (cf. Proof of theorem 4 - p 8)

REPRESENTATIONS OF GROUPS

- in a simplified way: abstract group → matrix (operator) group ⇒ easier to deal with
- provides many useful informations & tools even without explicit construction of the matrices
- repre need not to be faithful!

recall: • vector space V over the field K
- a set of objects with addition and scalar multiplication by an element from K

• field - set of elements with two binary operations $(+,\cdot)$, both commutative

(• division ring - only $+$ is commutative)

Def: Linear mapping between two vector spaces V and V' is a mapping $A: V \to V'$ satisfying

$$A(\alpha v + \beta w) = \alpha A(v) + \beta A(w) \quad \forall v, w \in V \ \& \ \alpha, \beta \in K$$

- both V & V' must be over the same field K

Def: Linear operator is a linear map $A: V \to V$

- $End(V)$ ... set of all lin. ops. on V (incl. $Ker\, A = 0$)
  ... endomorphisms on V
- $Aut(V)$ ... set of all invertible ($Ker\, A \neq 0$) lin. ops on V
  ... automorphismes on V
  $\sim GL(V)$, linear transformations on V

Def: Representation (ρ, V) of a group G on n-dim vekt. space V over the field K is homomorphism

$$\rho: G \to Aut(V)$$

- $\rho(g_1 g_2) = \rho(g_1)\rho(g_2)$, $\rho(e) = E_{Aut(V)}$, $\rho(g^{-1}) = \rho(g)^{-1}$

- notation
  - ρ is the mapping $G \to \mathrm{Aut}\, V$
  - $\rho(g) \equiv T(g) \in \mathrm{Aut}(V)$ is the lin. op. assigned to $g \in G$
  - $D(g)$ denotes the matrix in the case of matrix representation (i.e., the op. $T(g)$ expressed in a specific basis)

(*)

Def: Let $\{e_1, \dots, e_n\} \subset V$ be a basis of a n-dim. vect. space V. Then each automorphism is given by a matrix $D \in M^{n \times n}$ and, therefore, each $g \in G$ can be associated with a matrix

$$D(g) \in GL(n, K)$$

The mapping $D: G \to GL(n, K)$ is called <u>matrix representation</u> D of a group G

- V needs not be specified anymore
- matrix corresponding to an op. $T(g)$:

$$\boxed{D^i_j} \quad i \leftarrow \text{row}, \quad j \leftarrow \text{col.}$$

$x = e_i x^i$ ... e_i - basis vectors, x^i - coordinates

- $\boxed{T(g) e_i \equiv e_k D(g)^k_i}$ • note the order of e_k & $D(g)$: e_k is a "row vector"

- $x' = T(g) x = T(g) e_i x^i = e_k D(g)^k_i x^i = e_k x'^k$

$\Rightarrow \boxed{x'^k = D(g)^k_i x^i}$ • this is std. matrix-vector multiplication

- exercise: $D(e) = \mathbb{1}$ & $D(g_1 g_2) = D(g_1) D(g_2)$

$\Rightarrow D$ is a representation

(*)

Def: The vector space V is called the <u>representation space</u> of ρ and the dimension of V is the <u>dimension of the repre.</u>

Def: <u>Trivial representation</u> $\rho(g) = \mathbb{1}_{Aut(V)} \quad \forall g \in G$

- for $\dim V = 1$, triv. repre is called <u>totally symmetric irreducible representation</u>

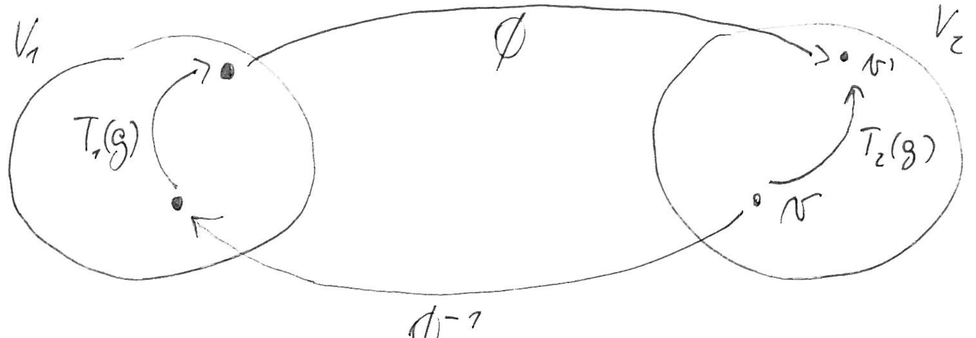
Def: If $\rho : G \to Aut(V)$ is injective, the <u>representation</u> (ρ, V) is called <u>faithful</u>

Notes:
- there $\exists$ infinitely many repre of a given G on vect. spaces of various dimensions
- even on the same V there might exist different repre's, many are, however, <u>equivalent</u>
- some of the multi-dimensional representations can be decomposed to several less-dimensional
  $\Rightarrow$ <u>reducibility</u>

Def: Let V_1 and V_2 be vector spaces. Two <u>representations</u> (ρ_1, V_1) and (ρ_2, V_2) of a group G are called <u>equivalent</u> if there exist an isomorphic mapping

$\phi : V_1 \to V_2$ such that

$$T_2(g)\, v = \phi \cdot T_1(g) \cdot \phi^{-1} v \quad \forall v \in V_2 \text{ and } \forall g \in G.$$



$$\boxed{\rho_1 \sim \rho_2}$$

Def: <u>Intertwining map</u> $S : V_1 \to V_2$ is a map that for two repre (ρ_1, V_1) & (ρ_2, V_2) satisfies

$$S \cdot T_1(g)\, v = T_2(g) \circ S v \quad \forall v \in V_1 \ \& \ \forall g \in G$$

Note: • $\rho_1 \sim \rho_2 \Leftrightarrow \exists S$ intertwining isomorphism ($\exists S^{-1}$)
• $\dim V_1 = \dim V_2$ is not sufficient condition for equivalence of ρ_1 & ρ_2

Examples • see homomorphisms $C_{2v} \to M^{3\times 3}$ & $C_{2v} \to (\{1,-1\},\cdot)$ discussed before

• $C_i = \{E, i\}$, $\rho : C_i \to Aut(\mathbb{R})$

1, $\rho_g(E) = \rho_g(i) = 1$ $\qquad g \leftrightarrow$ "gerade"

2, $\rho_u(E) = 1$, $\rho_u(i) = -1$ $\qquad$ "ungerade"

Theorem 11: Let two matrix representations D and $\tilde{D}$ of a group G be connected by a similarity transformation,

$$\tilde{D}(g) = B D(g) B^{-1} \quad \forall g \in G.$$

Then $D \sim \tilde{D}$.

Proof: • lets study basis transformation:

• $T(g)\, e_i = e_k D(g)^k_{\ i} \Rightarrow x'^k = D(g)^k_{\ i} x^i$

• basis transformation: $\hat{e}_i = e_j A^j_{\ i} \Leftrightarrow e_j = \hat{e}_i (A^{-1})^i_{\ j}$

$A : V \to V$ ... transformation matrix, $\det A \neq 0$

$$x = \hat{e}_i \hat{x}^i = e_j x^j = \hat{e}_i (A^{-1})^i_{\ j} x^j \Rightarrow \hat{x}^i = (A^{-1})^i_{\ j} x^j \equiv B^i_{\ j} x^j$$

$$\Rightarrow \boxed{\hat{x} = B x}$$

• $e_i \leftrightarrow$ repre $D(g)$: $x' = D(g)\, x$
$\hat{e}_i \leftrightarrow$ repre $\hat{D}(g)$: $\hat{x}' = \hat{D}(g)\, \hat{x}$ $\quad$ & $\hat{x}' = B x'$ for $\hat{x} = Bx$

$$\hat{x}' = \underbrace{\hat{D}(g)\, \hat{x}} = \hat{D}(g) B x = B x' = B D(g)\, x = \underbrace{B D(g) B^{-1} \hat{x}}$$

$$\Rightarrow \boxed{\hat{D}(g) = B D(g) B^{-1}}$$

- $A = B^{-1}$ defines isomorphism $V \to V \Rightarrow D$ & $\hat{D}$ are equivalent ☒

REDUCIBLE & IRREDUCIBLE REPRESENTATIONS

Def: Let $\phi: G \times V \to V$ be an action of a group G on a vect. space V and let $W \subset V$ is preserved under the action, that is,

$$T(g) w \in W \quad \forall g \in G \quad \forall w \in W \qquad (G \cdot W \subset W).$$

Then the <u>subspace</u> $W \subset V$ is called <u>invariant</u>.

Def: Inv. subspace W is called <u>irreducible</u> if it does not contain any other non-trivial invariant subspace.

Def: Let V contains an invariant subspace under the group action defined by a repre (ρ, V).
Then the <u>vect. space</u> V is called <u>reducible</u> and (ρ, V) is <u>reducible representation.</u>
<u>Representation</u> that is not reducible is called <u>irreducible.</u>

Def: A <u>subrepresentation</u> of a representation (ρ, V) is a representation $(\rho_{\restriction W}, W)$, where $W \in V$ invariant subspace under the group action def. by (ρ, V).

Reducible matrix representations

- let $W \subset V$ be an invariant subspace $\Rightarrow$ consider basis

$$W = \mathrm{span}(e_1, \dots, e_r) \qquad \text{span} \dots \text{ linear span}$$
$$W_\perp = V \setminus W = \mathrm{span}(e_{r+1}, \dots, e_d) \qquad \setminus \dots \text{ complement}$$

- W invariant $\Rightarrow$ $T(g)\, e_i = \sum_{k=1}^{r} e_k D(g)^k{}_i = \sum_k e_k D^W(g)^k{}_i$

$$\Rightarrow D(g) = \left(\begin{array}{c|c} D^W(g) & D^{WW_\perp}(g) \\ \hline 0 & D^{W_\perp}(g) \end{array}\right) \qquad (*)$$

- $D^W(g)$ forms a subrepre of $D(g)$:

$$D(g_1)D(g_2) = \left(\begin{array}{c|c} D^W(g_1)D^W(g_2) & \dots\text{exercise}\dots \\ \hline 0 & D^{W_\perp}(g_1)D^{W_\perp}(g_2) \end{array}\right)$$

- $D^{W_\perp}(g)$ is also repre of G, but it is not (in general) subrepre of G ... $W_\perp$ needs not be invariant

NOTE: • in an arbitrary basis of V, which does not conform to the structure of invariant subspaces, the matrices $D(g)$ do not have the block structure $(*)$

• however, $\exists$ similarity transform which will convert $D(g)$ to this form $\forall g \in G$ $\Downarrow$

Def: Matrix representation is said to be reducible if it is equivalent to a matrix repre $D(g)$, in which the matrices have block structure $(*)$ $\forall g \in G$.

Examples

a, 2-dim repre of $G(\mathbb{R}^+, \cdot)$

- $(\mathbb{R}^+, \cdot) \sim (\mathbb{R}, +)$: $x = e^y$ ... $x \in (\mathbb{R}^+, \cdot)$, $y \in (\mathbb{R}, +)$

$$x x' = e^{y+y'} \;;\; y = \log(x)$$

$$\Rightarrow D(x) = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \;:\; D(x) D(x') = \begin{pmatrix} 1 & y+y' \\ 0 & 1 \end{pmatrix} = D(x, x')$$

- $D(x)$ is reducible ; $W = \mathrm{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$

$D^W(x) = 1 \; \forall x$ ... triv. representation

b, 3-dim repre of $SO(2)$

- action of $SO(2)$ on $\mathbb{R}^3$: $T(g) v = R^z_\varphi v$

$$\left.\begin{array}{l} G \cdot \mathrm{span}(e_z) = \mathrm{span}(e_z) = W_1 \\ G \cdot \mathrm{span}(\{e_x, e_y\}) = \mathrm{span}(\{e_x, e_y\}) = W_2 \end{array}\right\} \text{both invariant}$$

$$R^z_\varphi = \begin{pmatrix} c\varphi & -s\varphi & 0 \\ s\varphi & c\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\nearrow$ $R^z_\varphi \downarrow W_1 = 1$ ... triv. repre

$\searrow$ $R^z_\varphi \downarrow W_2 = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$ ... faithful 2D repre

- $T(g) = R^x_\varphi \Rightarrow$ different decomposition to inv. subspaces

$(\{e_x\} \cup \{e_y, e_z\})$

Def: Repre (ρ, V) of a group G is called <u>completely reducible</u> if the repre. space V is a direct sum of irreducible invariant subspaces, $V = \bigoplus_i V_i$, under the group action

- <u>direct sum</u> of vect. spaces:

$$V = U \cup W \;\&\; U \cap W = \{0\} \Rightarrow V = U \oplus W$$

$\Rightarrow v = u + w$ is unique decomposition, $u \in U$ & $w \in W$

- in an appropriate basis complying with the decomposition the matrices of the representation have block-diag. form

$$D(g) = \begin{pmatrix} D^1(g) & & 0 \\ & D^2(g) & \\ 0 & & \ddots D^p(g) \end{pmatrix} \equiv \mathrm{diag}\left(D^1(g), D^2(g), \dots, D^p(g)\right)$$
$$\equiv \bigoplus_i D^i(g)$$

$\to D^i(g)$ are irreducible repre

<u>Def</u>: Matrix repre $D(g)$ is completely reducible if it is equivalent to a repre $D'(g)$ of block-diagonal matrices $\forall g \in G$.

<u>Examples</u>:

- $SO(2)$ on $\mathbb{R}^3$ is completely reducible after generalization to complex space:

$\xi_1 = e_x + i e_y \quad \xi_2 = e_x - i e_y$ generate 1D invariant subspaces:

NOTE (later): complex irred. repre of an abelian gr. are 1D

$$\begin{pmatrix} c\varphi & -s\varphi \\ s\varphi & c\varphi \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = e^{-i\varphi}\begin{pmatrix} 1 \\ i \end{pmatrix}, \dots$$

$$A = \frac{i}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \Rightarrow A^+\begin{pmatrix} c\varphi & -s\varphi \\ s\varphi & c\varphi \end{pmatrix} A = \begin{pmatrix} e^{-i\varphi} & 0 \\ 0 & e^{i\varphi} \end{pmatrix}$$

$\Rightarrow e^{-i\varphi}$ & $e^{i\varphi}$ are ~~two~~ non-equiv. faithful 1D repre

- $D(x) = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$ as a repre of $(\mathbb{R}^+, \cdot) \sim (\mathbb{R}, +)$ is not completely reducible:

Let $\exists A$: $\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = A \begin{pmatrix} f(y) & 0 \\ 0 & g(y) \end{pmatrix} A^{-1} \Rightarrow$ / det & Tr are invariant under similarity transf / $\Rightarrow$ $f(y)g(y) = 1$

$f(y) + g(y) = 2$

$\Rightarrow f(y) = g(y) = 1 \quad \forall$ (indep. of y)

Theorem XII

Any irreducible representation of a finite group is finite-dimensional.

Proof: • (ρ, V) is irrep, $x \in V$ arbitrary

$G\cdot x = \{T(g)x \,/\, g \in G\}$ is finite-dim set of vectors from V

$\Rightarrow$ span$(G\cdot x) \subset V$ is finite-dim invar. subspace of V

• (ρ, V) irrep $\Rightarrow$ span$(G\cdot x) = V \Rightarrow \dim V < +\infty$ ⊠

UNITARY REPRESENTATIONS

• Hilbert space:
- vect. space $\mathcal{H}$ with an inner (dot) product $\langle\cdot|\cdot\rangle$
- it is complete with respect to the metric induced by the inner product (Cauchy sequence is convergent within $\mathcal{H}$)
- it is separable (each $\psi \in \mathcal{H}$ is a limit of some sequence from a countable subset $M \subset \mathcal{H}$ $\Leftrightarrow$ $\exists$ countable basis)

• unitary operators:
- for bounded operators ($\exists K \in \mathbb{R}: \|U\psi\| \leq K\|\psi\| \;\forall \psi \in \mathcal{H}$)

  it is possible to define conjugation:

$$\langle \psi | A^+ \varphi \rangle \equiv \langle A\psi | \varphi \rangle \qquad \forall \psi, \varphi \in \mathcal{H}$$

- then for invertible bounded ops we can define unitarity:

$$\langle \psi | \varphi \rangle = \langle U\psi | U\varphi \rangle = \langle \psi | U^+ U \varphi \rangle \qquad \forall \psi, \varphi \in \mathcal{H}$$

$$\Leftrightarrow U^+U = \mathbb{1} \;\; \& \;\exists U^{-1} \Rightarrow U^{-1} = U^+ \Rightarrow UU^+ = \mathbb{1}$$

Def: Unitary repre of G is a repre on a Hilbert space $\mathcal{H}$ such that every $g \in G$ is represented by an unitary operator $U(g)$:

$$U(g)^+U(g) = U(g)U(g)^+ = \mathbb{1} \Leftrightarrow \langle U(g)\psi | U(g)\varphi \rangle = \langle \psi | \varphi \rangle$$
$$\forall g \in G \;\&\; \forall \psi, \varphi \in \mathcal{H}$$