Theorem XII

Any irreducible representation of a finite group is finite-dimensional.

Proof: • (ρ, V) is irrep, $x \in V$ arbitrary

$G \cdot x = \{T(g)x \,/\, g \in G\}$ is finite-dim set of vectors from V

$\Rightarrow$ span$(G \cdot x) \subset V$ is finite-dim invar. subspace of V

• (ρ, V) irrep $\Rightarrow$ span$(G \cdot x) = V \Rightarrow \dim V < +\infty$ $\boxtimes$

UNITARY REPRESENTATIONS

• Hilbert space:
- vect. space $\mathcal{H}$ with an inner (dot) product $\langle \cdot | \cdot \rangle$
- it is complete with respect to the metric induced by the inner product (Cauchy sequence is convergent within $\mathcal{H}$)
- it is separable (each $\psi \in \mathcal{H}$ is a limit of some sequence from a countable subset $M \subset \mathcal{H}$ $\Leftrightarrow$ $\exists$ countable basis)

• unitary operators:
- for bounded operators $(\exists K \in \mathbb{R} : \|U\psi\| \leq K\|\psi\| \;\; \forall \psi \in \mathcal{H})$

  it is possible to define conjugation:

$$\langle \psi | A^+ \varphi \rangle \equiv \langle A\psi | \varphi \rangle \qquad \forall \psi, \varphi \in \mathcal{H}$$

- then for invertible bounded ops we can define unitarity:

$$\langle \psi | \varphi \rangle = \langle U\psi | U\varphi \rangle = \langle \psi | U^+ U \varphi \rangle \qquad \forall \psi, \varphi \in \mathcal{H}$$

$$\Leftrightarrow U^+ U = \mathbb{1} \;\; \& \;\; \exists U^{-1} \Rightarrow U^{-1} = U^+ \Rightarrow UU^+ = \mathbb{1}$$

Def: Unitary repre of G is a repre on a Hilbert space $\mathcal{H}$ such that every $g \in G$ is represented by an unitary operator $U(g)$:

$$U(g)^+ U(g) = U(g) U(g)^+ = \mathbb{1} \Leftrightarrow \langle U(g)\psi | U(g)\varphi \rangle = \langle \psi | \varphi \rangle$$
$$\forall g \in G \;\; \& \;\; \forall \psi, \varphi \in \mathcal{H}$$

<u>Def:</u> <u>Matrix unitary repre</u> is such that every element $g \in G$ is represented by an unitary matrix
$$D(g)^{-1} = D(g)^{+}$$

<u>Theorem XIII</u>

Every finite-dimensional <u>reducible unitary</u> repre $(\rho, \mathcal{H})$ of a group G is <u>completely reducible</u>.

<u>Proof:</u> • ρ reducible $\Rightarrow \exists\, \mathcal{H}_1 \subset \mathcal{H}$ nontriv. inv. subspace

$\Rightarrow \mathcal{H}_1^{\perp} = \mathcal{H} \setminus \mathcal{H}_1$ is also invariant:

• $\psi \in \mathcal{H}_1^{\perp} \Rightarrow \langle \varphi | \psi \rangle = 0 \quad \forall \varphi \in \mathcal{H}_1$

• $\mathcal{U}(g)\psi \in \mathcal{H}_1^{\perp}$: $\langle \varphi | \mathcal{U}(g)\psi \rangle = \langle \mathcal{U}(g)\mathcal{U}(g)^{-1}\varphi | \mathcal{U}(g)\psi \rangle$

$$= \langle \mathcal{U}(g^{-1})\varphi | \mathcal{U}(g)^{+}\mathcal{U}(g) \rangle = \underset{\mathcal{H}_1 \text{ inv.} \Rightarrow \mathcal{U}(g^{-1})\varphi \in \mathcal{H}_1}{\langle \varphi' | \psi \rangle} = 0$$

• if $\mathcal{H}_1$ or $\mathcal{H}_1^{+}$ are further reducible then the same decomposition can be repeated until complete reducibility ... provided $\dim \mathcal{H} < +\infty$ ⊠

<u>Note:</u> 2D-repre $(\mathbb{R}^{+}, \cdot)$ is not unitary $\Rightarrow$ Th. does not apply

<u>Theorem XIV</u>

Every finite-dim representation of a finite or compact Lie group is equivalent to some unitary representation.

<u>NB:</u> we do not require the repre space to be a Hilbert space!

Proof (hint):

- every finite-dim vect. space $\sim \mathbb{R}^n$ or $\mathbb{C}^n$
  $\Rightarrow$ it is possible to select basis $\{e_1, \dots, e_n\}$

  $\Rightarrow X = x^i e_i \quad \forall x \in V$

  $\Rightarrow \langle x | y \rangle \equiv (x^i)^* y^i$ is a legal inner product:

  - $\langle x | y \rangle = \langle y | x \rangle^*$ (conjugate sym.)
  - linear in first argument
  - $\langle x | x \rangle > 0 \quad \forall x \in V \setminus \{0\}$ (positive definite)

- if (ρ, V) is not unitary with resp. to $\langle \cdot | \cdot \rangle$, it is possible to construct another inner product

$$(x|y) \equiv \frac{1}{\# G} \sum_{g'} \langle T(g') x \,|\, T(g') y \rangle$$

$\Rightarrow$ /rearr. theorem/ $\Rightarrow \quad (T(g)x \,|\, T(g)y) = (x|y)$

$\Rightarrow (\rho, V)$ is unitary with resp. to $(\cdot | \cdot)$ $\boxtimes$

Note: • $(\cdot|\cdot)$ is equivalent to $\langle \cdot | \cdot \rangle$ - induces equal topology (ie., open sets on V) $\Leftrightarrow$ the same convergent sequences

$\Leftrightarrow \exists a, b \in \mathbb{R},\ 0 < a \leq b: \quad a \langle x|x \rangle \leq (x|x) \leq b \langle x|x \rangle \quad \forall x \in V$

- compact Lie groups $\equiv$ compact smooth manifolds
  $\equiv$ parametrized by coordinates from a compact subspace of $\mathbb{R}^n$ if $\exists$ global map;
  (compact set in $\mathbb{R}^n$: closed & bounded)

  - $SO(2)$, $O(n)$ are compact, $(\mathbb{R}^+, \cdot)$ is not
- for comp. Lie groups $\exists$ left-invariant measure
  $\exists \int_G dg < +\infty : \quad \int_G f(hg)\, dg = \int_G f(g)\, dg \quad \Rightarrow \frac{1}{\# G} \sum_g \to \frac{1}{|G|} \int_G dg$

Theorem XV (Maschke)

Every finite-dim reducible repre of a finite (compact Lie) group is completely reducible.

Proof: • combine Th XIV & XIII $\boxtimes$

SCHUR'S LEMMAS

Lemma SL1:

Let (ρ_1, V_1) and (ρ_2, V_2) be irreducibilni repre of a group G & let S is the intertwining mapping

$$S: V_1 \to V_2 \qquad S\,T_1(g)\,v_1 = T_2(g)\,S\,v_1 \quad \forall g \in G \;\; \forall v_1 \in V_1$$

Then either $S = 0$ ($\Leftrightarrow \operatorname{Ker} S = V_1$) or S is isomorphic map & $\rho_1 \simeq \rho_2$.

Proof: • $\operatorname{Ker} S$ & $\operatorname{Im} S$ are invariant subspaces of V_1 & V_2, resp:

a) $v_1 \in \operatorname{Ker} S \Rightarrow S\,T_1(g)\,v_1 = T_2(g)\,S\,v_1 = 0 \Rightarrow T_1(g)\,v_1 \in \operatorname{Ker} S$

b) $v_2 \in \operatorname{Im} S \Rightarrow \exists v_1 \in V_1 : v_2 = S\,v_1$

$$\Rightarrow T_2(g)\,v_2 = T_2(g)\,S\,v_1 = S\,T_1(g)\,v_1 = S\,v_1' \Rightarrow T_2(g)\,v_2 \in \operatorname{Im} S$$

• V_1 & V_2 are irreducible $\Rightarrow$ there are only 2 options:

a) $\operatorname{Ker} S = V_1$ & $\operatorname{Im} S = \{0\} \Leftrightarrow S = 0$

b) $\operatorname{Im} S = V_2$ & $\operatorname{Ker} S = \{0\} \Rightarrow S$ is bijective:

• surjective by $\operatorname{Im} S = V_2$

• injective: $S\,v_1 = S\,v_1' = v_2 \Rightarrow S(v_1 - v_1') = 0$

$$\Rightarrow v_1 - v_1' \in \operatorname{Ker} S \Rightarrow v_1 - v_1' = 0$$

$\boxtimes$

<u>Lemma SL2</u> (consequence of SL1 for finite-dim irreps)

Let (ρ, V) be a <u>complex</u> <u>finite-dim</u> irreducible repre of a group G and S is intertwining operator on V ($S: V \to V$) such that

$$T(g) S v = S T(g) v \quad \forall g \in G \ \forall v \in V.$$

Then $S = \lambda \mathbb{1}$ for some $\lambda \in \mathbb{C}$.

<u>Proof:</u> • let $S \neq 0 \Rightarrow \exists \lambda \in \mathbb{C} \ \exists v_\lambda \in V: S v_\lambda = \lambda v_\lambda$

(in a finite dim there always exists a solution of a characteristic polynomial in $\mathbb{C} \Rightarrow$ every operator has an eigenvalue)

• eigen-subspace $V_\lambda \subset V$ corresponding to λ is invariant under the action of G:

$$v \in V_\lambda \Rightarrow S T(g) v = T(g) S v = \lambda T(g) v \Rightarrow T(g) v \in V_\lambda$$

$$\Rightarrow /(\rho, V) \text{ irred.}/ \Rightarrow V_\lambda = V \Rightarrow S = \lambda \mathbb{1} \quad (S e_i = \lambda e_i \ \forall e_i \in \text{basis})$$

<u>Note:</u> • in an infinite-dim V the subspace V_λ need not to be closed:

$\{v_k\} \subset V_\lambda$ Cauchy sequence in V_λ does not imply $\lim_{k \to \infty} v_k = v \in V_\lambda \Rightarrow ST(g) = T(g) S$ does not imply invariance of V_λ

CONSEQUENCES OF SCHUR LEMMAS

Theorem XVI

Complex finite-dim irreducible representations of an Abelian group are one-dimensional.

Proof: • (ρ, V) finite-dim, G Abelian $\Rightarrow$

$$T(g)T(h) = T(h)T(g) \quad \forall g,h \in G$$

$$\stackrel{(SL2)}{\Longrightarrow} T(h) = \lambda(h)\,\mathbb{1} \quad \forall h \in G$$

$\Rightarrow \rho$ is either reducible or 1-dim. $\boxtimes$

Note: $SO(2)$ is Abelian but we had to move into complex repre space to obtain 1-dim irreps.

Theorem XVII (Orthogonality relations for matrix representations)

Let D^μ and D^ν are two unitary irreducible matrix repre of a finite (compact Lie) group. Dimensions of the representations are d_μ & d_ν, resp. Let D^μ and D^ν are not equivalent for $\mu \neq \nu$ or identical for $\mu = \nu$. Then

$$\sum_g [D^\mu(g)^*]^j_i \, D^\nu(g)^k_l = \frac{\#G}{d_\mu} \delta_{\mu\nu}\, \delta_{kj}\, \delta_{il} \qquad (*)$$

Proof: • B is an arbitrary $d_\mu \times d_\nu$ matrix

$$\Rightarrow A = \sum_g D^\mu(g^{-1}) B D^\nu(g) \quad \Rightarrow D^\mu(h) A = A D^\nu(h) \quad \forall h \in G:$$

$$\sum_g D^\mu(h) D^\mu(g^{-1}) B D^\nu(g) = / hg^{-1} \equiv g'^{-1} \Rightarrow g' = gh^{-1} \text{ \& rearr. theorem} /$$

$$= \sum_{g'} D^\mu(g') B D^\nu(g'h) = A D^\nu(h)$$

1, D^μ not equiv to D^ν $\Rightarrow /SL2/ \Rightarrow$ $A \equiv 0$ & choose

$B^j_k = \delta_{jr}\delta_{ks}$ for a fixed r,s

$$\Rightarrow 0 = \sum_{jk}\sum_g D^\mu(g^{-1})^i_j \, \delta_{jr}\delta_{ks} \, D^\nu(g)^k_\ell = \sum_g D^\mu(g^{-1})^i_r \, D^\nu(g)^s_\ell$$

$$\boxed{0 = \sum_g \left[D^\mu(g)^r_i\right]^* D^\nu(g)^s_\ell \quad \text{for } \mu \neq \nu}$$

2, $D^\mu \sim D^\nu \Rightarrow \exists S: \; D^\nu(g) = S D^\mu(g) S^{-1}$

$$\Rightarrow D^\mu(h) A = A S D^\mu(h) S^{-1} \rightarrow D^\mu(h) AS = AS D^\mu(h)$$

$$\Rightarrow /SL2/ \Rightarrow \; AS = \lambda \mathbb{1}_{d_\mu \times d_\mu}$$

- λ from Tr: $Tr(AS) = d_\mu \lambda = Tr\left(\sum_g D^\mu(g^{-1}) B S D^\mu(g) S^{-1} S\right)$

$$\Rightarrow \lambda = \frac{\#G}{d_\mu} Tr(BS) = /B^j_k = \delta_{jr}\delta_{ks}/ = \frac{\#G}{d_\mu} B^j_k S^k_j$$

$$\Rightarrow \boxed{\lambda = \frac{\#G}{d_\mu} S^s_r}$$

- $$A^i_\ell = \lambda (S^{-1})^i_\ell = \sum_{jk}\sum_g D^\mu(g^{-1})^i_j \, \delta_{jr}\delta_{ks} \, D^\nu(g)^k_\ell$$

$$\Rightarrow \sum_g \left(D^\mu(g)^r_i\right)^* D^\nu(g)^s_\ell = \frac{\#G}{d_\mu} S^s_r (S^{-1})^i_\ell$$

3, $\mu = \nu \Rightarrow S^i_\ell = (S^{-1})^i_\ell = \delta_{i\ell}$

$$\Rightarrow \sum_g \left(D^\mu(g)^r_i\right)^* D^\nu(g)^s_\ell = \frac{\#G}{d_\mu} \delta_{\mu\nu}\delta_{sr}\delta_{i\ell} \quad \boxtimes$$

Direct consequence: $\sum_\mu d_\mu \leq \#G$:

(*) is orthogonality relation between $\#G$-dim vectors $(D^\mu(g_1)^i_j, \ldots, D^\mu(g_{\#G})^i_j)$, which are indexed by (μ, i, j)

$\Rightarrow$ total muber of these orthog. vectors is $\sum_\mu d_\mu^2$ & must be $\leq \#G$

CHARACTER OF A REPRESENTATION

- goal: find a property which will be equal for equiv. repre's and, if possible, different for non-equiv repre's (the second goal will be met only for finite groups)

$\Rightarrow$ for matrix repre, we are looking for invariants with resp. to similarity transforms

a, all eigenvalues
b, Tr

Def: Let (ρ, V) be a repre of a gr. G on a finite-dim V and let D be a corresp. matrix repre in some basis. Then the function $\chi : G \to \mathbb{C}$

$$g \mapsto \operatorname{Tr} D(g)$$

is called character of a representation (also character system) and the number $\chi(g) = \operatorname{Tr} D(g)$ is character of an element $g \in G$ in a repre (ρ, V).

Notes:

- $\chi(g') = \chi(g) \quad \forall g' \in (g) \qquad \Leftarrow \operatorname{Tr}(ABC) = \operatorname{Tr}(CAB)$
  $\Rightarrow$ character of an element is character of the whole class
- χ is equal for all equiv. repre's $\Rightarrow$ it is a characteristics of the equivalence class
- equal χ does not imply equiv. repre:
  $(\mathbb{R}^+, \cdot): \quad D(y) = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$ not equiv. to $\hat{D}(y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- for finite groups, we will show $\chi = \chi' \Rightarrow D \sim D'$
- $\chi(e) = \dim V \qquad (\Leftarrow D(e) = \mathbb{1})$
- $\chi(g^{-1}) = \chi(g)^*$ for finite-dim repre ($\Leftarrow$ Th. XIV & equivalence to a unitary repre.)

- character of a reducible representation (block-diag. matrices) completely

$$\boxed{D(g) = \bigoplus_i D^i(g) \Rightarrow \chi(g) = \sum_i \chi^i(g)}$$

Theorem XVIII (orthog. relations for χ)

Let χ^μ and χ^ν be characters of two IRREPs of a finite (or compact Lie) group on a complex finite-dim vect. spaces and let the IRREPs are non-equiv for $\mu \neq \nu$.

Then

$$\boxed{\sum_g \chi^\mu(g^{-1})\,\chi^\nu(g) = \sum_g \chi^\mu(g)^* \chi^\nu(g) = \#G\,\delta_{\mu\nu}}$$

- for compact Lie groups: $\sum_g \to \int_G dg$

Proof: directly from Th. XVII:

- $\sum_g D^\mu(g^{-1})^i_j\, D^\nu(g)^k_l = \frac{\#G}{d_\mu}\delta_{\mu\nu}\,\delta_{il}\,\delta_{kj} \Rightarrow / i=j \;\&\; k=l,\ \sum_{ik} / \Rightarrow$

$$\Rightarrow \sum_g \chi^\mu(g^{-1})\,\chi^\nu(g) = \frac{\#G}{d_\mu}\delta_{\mu\nu}\sum_{ki}(\delta_{ik})^2 = \#G\,\delta_{\mu\nu}$$

- $\chi^\mu(g^{-1}) = \chi^\mu(g)^*$: ρ^μ ~ unit. repre $\Rightarrow \lambda_i^{-1} = \lambda_i^*$ for all eigenvalues (eigenvalues of unitary op/matrix are complex units: $|\lambda_i| = 1$)

$\boxtimes$

Note:

- $\chi(g') = \chi(g)\ \ \forall g' \in (g) \Rightarrow$ orthog. relations can be written using characters of classes:

$$\boxed{\sum_{k=1}^{N_c} n_k\, \chi^\mu(g_k)^*\, \chi^\nu(g_k) = \#G\,\delta_{\mu\nu}} \qquad (+)$$

- $k = 1, \dots, N_c$ numbers all distinct classes (g_k)
- $n_k = \#(g_k)$

- direct consequence: $\boxed{\# \text{IRREPs} \leq N_c}$

  ⇐ (+) says that characters of non-equiv. IRREPs form a set of orthog. vectors in N_c-dim vect. space

- we will prove later that there is in fact "="

Theorem XIX:

Let G be a finite or compact Lie group. Then equality of characters of two representations is sufficient condition for their equivalence.

Proof: 1) let ρ^μ and ρ^ν are two non-equiv IRREPs with equal characters

$$\overset{\text{non}}{\underset{\text{equiv}}{\Rightarrow}} \sum_g \chi^\mu(g)^* \chi^\nu(g) = 0 \quad \& \quad \chi^\mu(g) = \chi^\nu(g) \; \forall g$$

$$\Rightarrow \sum_g \chi^\mu(g)^* \chi^\mu(g) = \#G \quad \text{↯}$$

2) let ρ^i & ρ^j are reducible ⇒ /Maschke/

→ they are completely reducible

$$\Rightarrow \rho^i = \bigoplus_\alpha n^i_\alpha \rho^\alpha \qquad \rho^j = \bigoplus_\alpha n^j_\alpha \rho^\alpha$$

- (here ρ^α are all IRREPs of G, we already know it is a finite expansion)

- $\rho^i = \bigoplus_\alpha n^i_\alpha \rho^\alpha$ means that the repre space V^i contains n^i_α-times repre space V^α as a invar. subspace

  → in terms of matrices:

$$D^i = \text{diag}(\ldots, \underbrace{D^\alpha, D^\alpha, \ldots, D^\alpha}_{n^i_\alpha\text{-times}}, \ldots)$$

$$\Rightarrow \chi^i(g) = \sum_\alpha n^i_\alpha \chi^\alpha(g) \qquad \chi^j(g) = \sum_\alpha n^j_\alpha \chi^\alpha(g)$$

- by asumption, $\chi^i(g) = \chi^j(g) \quad \forall g$

$$\to \sum_\alpha (n^i_\alpha - n^j_\alpha)\, \chi^\alpha(g) = 0 \quad \forall g \qquad / \cdot \chi^\beta(g)^*, \sum_g$$

$$\Rightarrow 0 = \sum_\alpha (n^i_\alpha - n^j_\alpha) \sum_g \chi^\beta(g)^* \chi^\alpha(g) = \sum_\alpha (n^i_\alpha - n^j_\alpha)\, \#G\, \delta_{\alpha\beta}$$

$$= (n^i_\beta - n^j_\beta)\#G \Rightarrow n^i_\beta = n^j_\beta \quad \forall \beta$$

$\to \rho^i$ & ρ^j has the same decomposition to IRREPs

$\Rightarrow \rho^i \simeq \rho^j$ $\boxtimes$

<u>Note:</u> • we have prooved also

<u>Theorem XX:</u> Let (ρ, V) be reducible repre of a finite (complex Lie) G with decomposition to IRREPs

$$\rho = \bigoplus_\alpha n_\alpha \rho^\alpha$$

Then

$$n_\alpha = \frac{1}{\#G} \sum_g \chi^\alpha(g)^* \chi(g)$$

<u>Note:</u> • Th XX implies that the decomposition $\rho = \bigoplus_\alpha n_\alpha \rho^\alpha$ is unique

IRREDUCIBLE REPRESENTATIONS OF FINITE GROUPS

- remember: we know there is only a finite number of them:
  - #IRREPs ≤ number of distinct classes
  - $\sum_\mu d_\mu^2 \le \#G$

Def: Regular representation of a finite G is

$$D^r(g_s)^k_l = \begin{cases} 1 & \text{for } g_s g_l = g_k \\ 0 & \text{otherwise} \end{cases}$$

- $\dim D^r = \#G$
- in each row and each col. there is exactly one 1 (rearr.)
- it is indeed a repre:

a) $g_s = e \Rightarrow 1$ for $g_l = g_k \Rightarrow D^r(e) = \mathbb{1}$

b) $\sum_l D^r(g_r)^k_l \, D^r(g_s)^l_m = \big/ g_r g_l = g_k \ \& \ g_s g_m = g_l$

$\Rightarrow g_r g_s g_m = g_k \big/ = D^r(g_r g_s)^k_m$ ✓

- character of D^r:
  - $\chi(e) = \#G$
  - $\chi(g \neq e) = 0$ (non-zero elements only off-diagonal: $g_s g_k = g_k \Rightarrow g_s = e$)

Example:

e	a	b
a	b	e
b	e	a

$$\Rightarrow D^r(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad D^r(b) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$D^r(a) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Theorem XXI: $\sum_{\mu} d_\mu^2 = \#G$ (sum over non-equiv. IRREPs)

Proof: • Maschke $\Rightarrow D^r$ is completely reducible

$$\Rightarrow D^r = \bigoplus_{\mu} n_\mu^r D^\mu$$

$$\Rightarrow n_\mu^r = \frac{1}{\#G} \sum_g \chi^\mu(g)^* \chi^r(g) = \frac{1}{\#G} \underbrace{\chi^\mu(e)^*}_{= d_\mu} \chi^r(e) = d_\mu$$

$$\Rightarrow D^r = \bigoplus_{\mu} d_\mu D^\mu \Rightarrow \chi^r(g) = \sum_\mu d_\mu \chi^\mu(g) \quad / g = e$$

$$\Rightarrow \#G = \sum_\mu d_\mu^2 \qquad \boxtimes$$

• next we want to proove $\#IRREP_s = \#(\text{classes})$

Lemma:

Let C be a set of elements from G (each element can be incl. multiple - times).

Then $gCg^{-1} = C \quad \forall g \in G \Leftrightarrow C = \sum_{(g_k)} a_k (g_k)$.

• C need not be a group
• $\sum_{(g_k)}$ runs over distinct classes
• $a_k \geq 0 \Rightarrow$ elements of G might be included multiple-times in C
• note: $H \triangleleft G \Rightarrow H$ consists of complete classes.

Proof: $\Leftarrow$ | $g(g_k)g^{-1} = (g_k)$ (rearr.) : $g(h g_k h^{-1})g^{-1} = \hat{h} g_k \hat{h}^{-1}$

$\Rightarrow$ | • $gCg^{-1} = C$ & let $C = \sum_{(g_k)} a_k(g_k) + R$ such that $\exists a \in R : (a) \notin R$

• since $g(g_k)g^{-1} = (g_k) \; \forall (g_k) \; \forall g \in G \Rightarrow gRg^{-1} = R \; \forall g \in G$