RELATIONS BETWEEN REPRESENTATIONS OF A GROUP & ITS SUBGROUPS

A) SUBDUCED REPRESENTATIONS

Def: Let $T(g)$ be operators of a repre (ρ, V) of G and let $H < G$ is a subgroup. Then

$$\rho_{\downarrow H} = \{T(h) \mid h \in H\}$$

forms subduced representation of H

- $\rho_{\downarrow H}$ in general reducible even for ρ IRREP of G

$$\rho_{\downarrow H} = \bigoplus_{\mu} \alpha_{\mu}^{\downarrow H} \rho_H^{\mu} \Leftrightarrow \alpha_{\mu}^{\downarrow H} = \frac{1}{\# H} \sum_{h \in H} \chi_H^{\mu}(h)^* \chi(h)$$

Example: • $H = C_s = \{E, \sigma_v\} < G = C_{3v} = \{E, 2C_3, 3\sigma_v\}$

C_{3v}	E	$2C_3$	$3\sigma_v$
A_1	1	1	1
A_2	1	1	-1
E	2	-1	0

$\to A_{1\downarrow C_s} = A'$

$\to A_{2\downarrow C_s} = A''$

$\to E_{\downarrow C_s} = A' \oplus A''$

C_s	E	σ_v
A'	1	1
A''	1	-1

B) INDUCED REPRESENTATIONS (For finite groups)

- let $H < G$ & $D_H(h)$ is a d-dim. repre of H

$\Rightarrow$ can we construct a repre of the full G?

YES, by explicit construction of its basis!

1, decomposition of G into left cosets with resp. to H

$$G = p_1 H + p_2 H + \dots + p_M H \qquad M = \frac{\# G}{\# H} \quad \text{(Lagrange)}$$

- p_i fixed representatives of individual classes
- $p_1 = e$

2, basis of the induced repre of G

- let $\{\phi_1, \dots, \phi_d\}$ be the basis of D_H

$$\Rightarrow T(h)\phi_i = \sum_{j=1}^{d} \phi_j \, D_H(h)^j_{\ i}$$

$$\operatorname{span}\{\phi_{ti}\} = \bigoplus_{t=1}^{M} \underbrace{p_t V_H}_{\text{isomorphic copy of } V_H}$$

repre space of D_H

$\Uparrow$

- def. $\boxed{\phi_{ti} \equiv T(p_t)\phi_i \qquad t = 1,\dots,M;\ i = 1,\dots,d}$

(they are abstract objects not living in the repre space of D_H, we don't really know what they are...)

$\Rightarrow \phi_{ti}$ forms a basis of dM-dim repre of G

(we will not prove they are lin. indep.):

$$T(g)\phi_{ti} = T(g p_t)\phi_i = T(p_s p_s^{-1} g p_t)\phi_i = T(p_s) T(p_s^{-1} g p_t)\phi_i$$

- p_s: $\quad p_s^{-1} g p_t \in H \Leftrightarrow g p_t \in p_s H$

(such p_s exists and is unique because $g p_t = g' \in G$ and each element of G belongs to exactly one coset)

$$\Rightarrow T(g)\phi_{ti} = T(p_s) \sum_j \phi_j D_H(p_s^{-1} g p_t)^j_{\ i} = \sum_j \phi_{sj} D_H(p_s^{-1} g p_t)^j_{\ i}$$

for $g p_t \in p_s H$

- def. $\boxed{\delta_{st}(g) = \begin{cases} 1 & g p_t \in p_s H \\ 0 & g p_t \notin p_s H \end{cases}}$

$$\Rightarrow \boxed{D_G(g)^{sj}_{\ ti} = \delta_{st}(g) \, D_H(p_s^{-1} g p_t)^j_{\ i}}$$

induced repre of G

$D_G = D_{H \uparrow G}$

- note that we don't really know what $T(p_t)\phi_i$ is but we don't need to in the end...

3, $D_G(g)^{sj}_{ti}$ is indeed a repre

a, $g=e \Rightarrow \delta_{st}(e) = \delta_{st}$: $ep_t = p_t \in p_t H$

$$\Rightarrow D_H(p_s^{-1} g p_t) = D_H(p_t^{-1} e p_t) = D_H(e) = \mathbb{1}$$

$$\Rightarrow D_G(e) = \mathbb{1}_{dM \times dM}$$

this selects one unique r from the $\sum_r$

b, $$\sum_{rk} D_G(g)^{sj}_{rk} D_G(g')^{rk}_{ti} = \sum_{rk} \delta_{sr}(g)\,\delta_{rt}(g')\,D_H(p_s^{-1} g p_r)^j_k D_H(p_r^{-1} g' p_t)^k_i$$

$= /$ $gp_r = p_s h$ & $g' p_t = p_r h' \Rightarrow gg' p_t = g p_r h' = p_s h h' \Leftrightarrow \delta_{st}(gg')$

& $\exists!\, p_r$ such that $gp_r \in p_s H \Rightarrow$ sum over r gives just single nonzero contrib. /

$$= \delta_{st}(gg')\, D_H(p_s^{-1} gg' p_t)^j_i \quad \boxtimes$$

<u>Theorem:</u> D_H unitary $\Rightarrow D_{G\uparrow H}$ unitary

<u>Proof:</u> $$[D_G(g)^{-1}]^{sj}_{ti} = [D_G(g^{-1})]^{sj}_{ti} = \delta_{st}(g^{-1})\, D_H(p_s^{-1} g^{-1} p_t)^j_i$$

$= /$ $g^{-1} p_t = p_s h \Leftrightarrow g p_s = p_t h' \Rightarrow \delta_{st}(g^{-1}) = \delta_{ts}(g)$ /

$$= \delta_{ts}(g)\, D_H((p_t^{-1} g p_s)^{-1})^j_i = /\text{unitarity}/ = \delta_{ts}(g) [D_H(p_t^{-1} g p_s)^i_j]^*$$

$$= [D_G(g)^\dagger]^{sj}_{ti} \quad \boxtimes$$

- character of an induced repre

$$\chi_G(g) = \sum_{sj} \delta_{ss}(g)\, D_H(p_s^{-1} g\, p_s)^j_j = \sum_s \delta_{ss}(g)\, \chi_H(\underset{H\,!}{\underset{\uparrow}{p_s^{-1} g\, p_s}})$$

- summation over M selected elements p_s can be replaced by a sum over $\forall g' \in G$ with additional condition $g'^{-1} g g' \in H$ $\Rightarrow$ instead of a single representative of each coset we take every element from the coset; i.e., $\# H$ equal contributions instead of 1:

$$g\, p_s \in p_s H \;(\delta_{ss}(g)) \;\&\; g' \in p_s H \Rightarrow g'^{-1} g\, g' = (p_s h')^{-1} g\, (p_s h)$$
$$= h'^{-1} p_s^{-1} g\, p_s h' = h'^{-1} h\, h' \in H \Rightarrow g g' \in g' H = p_s H \checkmark$$

$$\Rightarrow \boxed{\chi_G(g) = \sum_s \delta_{ss}(g)\, \chi_H(p_s^{-1} g\, p_s) = \frac{1}{\# H} \sum_{\substack{g' \\ g'^{-1} g g' \in H}} \chi_H(g'^{-1} g\, g')}$$

- both expressions useful – in different situations

- decomposition of $D_{H\uparrow G} = D_G$

- assume we are inducing IRREP of H: $\rho_G^{\nu\uparrow G}$ is a repre of G induced from the ρ_H^ν IRREP of H:

$$\rho_G^{\nu\uparrow G} = \bigoplus_\mu \alpha_\mu^{\nu\uparrow G} \rho_G^\mu \qquad \rho_H^{\mu\downarrow H} = \bigoplus_\nu \alpha_\nu^{\mu\downarrow H} \rho_H^\nu$$

Theorem XXV: (Frobenius)

$$\boxed{\alpha_\mu^{\nu\uparrow G} = \alpha_\nu^{\mu\downarrow H}}$$

in words: IRREP ρ_G^μ of G is contained in $\rho_G^{\nu\uparrow G}$ as many times as is the IRREP ρ_H^ν contained in $\rho_H^{\mu\downarrow H}$,

very easy to determine!

$$\Rightarrow \chi_G^{\nu\uparrow G}(g) = \sum_\mu \alpha_\mu^{\nu\uparrow G} \chi_G^\mu(g) = \sum_\mu \alpha_\nu^{\mu\downarrow G} \chi_G^\mu(g)$$

Proof: $\alpha_\mu^{\nu\uparrow G} = \frac{1}{\#G} \sum_g \chi_G^\mu(g)^* \frac{1}{\#H} \sum_{\substack{g'\in G \\ g'^{-1}gg'\in H}} \chi_H^\nu(g'^{-1}gg') =$

$$= \Big/ \begin{array}{l} h = g'^{-1}gg' \Rightarrow g = g'hg'^{-1} \\ \Rightarrow \sum_g \sum_{g'} = \sum_h \sum_{g'} \end{array} \Big/ = \frac{1}{\#G\#H} \sum_{h\in H} \chi_H^\nu(h) \sum_{g'} \underbrace{\chi_G^\mu(g'hg'^{-1})^*}_{\#G\times \text{ the same!}}$$

$$= \frac{1}{\#H} \sum_{h\in H} \chi_H^\nu(h)\, \chi_G^\mu(h)^* = \left(\alpha_\nu^{\mu\downarrow H}\right)^* \underset{\mathbb{N}_0}{\overset{\uparrow}{=}} \alpha_\nu^{\mu\downarrow H}$$

$\boxtimes$

Examples: 1) $H = \{e\}$, D_H is trivial repre $\Rightarrow$ $D_{H\uparrow G}$ is regular repre

SYMMETRIES IN QM

- G is a symmetry group of a system
  $\Leftrightarrow$ $\hat{H}$ (Schr. eq.) invariant with resp. to symmetry operations from G

  $\rightarrow$ what does it mean?
  $\rightarrow$ how is the theory of repre useful?

- description of q. system – vector from Hilbert space $\mathcal{H}$
  $\rightarrow$ 1 spinless particle $\Rightarrow$ $\psi \in L^2(\mathbb{R}^3)$ (bound states)
- action of G on $\mathcal{H}$ $\Rightarrow$ typically ∞-dim unitary representation

$$\rho : G \rightarrow ISO(\mathcal{H})$$

- non-relativistic QM: G typically $O(3), SO(3)$, point groups, crystalographic groups

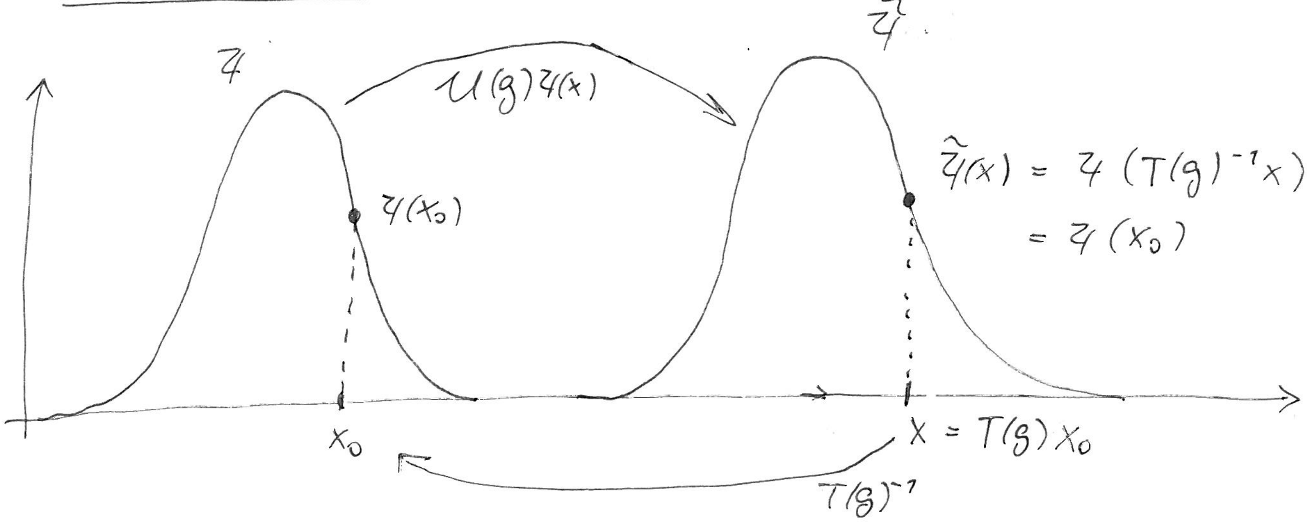
Action of G on $L^2(\mathbb{R}^3)$

1/ action of G on $\mathbb{R}^3$: $g \mapsto T(g) \in Aut(\mathbb{R}^3)$

$$\boxed{x' = T(g)x}$$ ... coordinate transformation

2/ corresp. unitary op. $\mathcal{U}(g) \in ISO(L^2(\mathbb{R}^3))$:

$$\boxed{\tilde{\psi}(x) = \mathcal{U}(g)\psi(x) \equiv \psi(T(g)^{-1}x) \quad \forall \psi \in L^2(\mathbb{R}^3)}$$



- $\mathcal{U}(g)$ is indeed a representation of G:

$$\mathcal{U}(g_1)\mathcal{U}(g_2)\psi(x) = \mathcal{U}(g_1)\psi(T(g_2)^{-1}x) = \psi(T(g_2)^{-1}T(g_1)^{-1}x)$$

note the order, $T(g_1)$ acts directly on x

$$= \psi([T(g_1)T(g_2)]^{-1}x) = \psi(T(g_1g_2)^{-1}x) = \mathcal{U}(g_1g_2)\psi(x) \qquad \boxtimes$$

Transformation of operators

- $\psi \mapsto \mathcal{U}(g)\psi \Rightarrow A \mapsto \hat{A}$ such that matrix elements remain unchanged

→ this must hold for any $\mathcal{U}(g)$ generated by coordinate transformation, regardless symmetry - shifting/rotating whole system in space can't change physics

$$\Rightarrow \langle\psi|A|\varphi\rangle = \langle\mathcal{U}(g)\psi|\hat{A}|\mathcal{U}(g)\varphi\rangle$$

$$\overset{\text{unitarity}}{=} \langle\psi|\mathcal{U}(g)^{+}\hat{A}\,\mathcal{U}(g)|\varphi\rangle$$

$$\Rightarrow \boxed{\hat{A} = \mathcal{U}(g)\,A\,\mathcal{U}(g)^{+}}$$

- what does it mean in "x-representation":

$$\phi(x) = A(x)\psi(x) \xrightarrow{\mathcal{U}(g)} \phi(T(g)^{-1}x) = A(T(g)^{-1}x)\,\psi(T(g)^{-1}x)$$

$$\phi(T(g)^{-1}x) = \mathcal{U}(g)\,A(x)\,\psi(x) = A(T(g)^{-1}x)\,\mathcal{U}(g)\psi(x)$$

$$= \hat{A}(x)\,\mathcal{U}(g)\psi(x)$$

$$\Rightarrow \boxed{\hat{A}(x) = \mathcal{U}(g)\,A\,\mathcal{U}(g)^{+} = A(T(g)^{-1}x)}$$

Note: • multi-component ψ (bispinors, ...) ... individual components might mix under action of G ⇒

$$\mathcal{U}(g)\begin{pmatrix}\psi_1(x)\\ \vdots\\ \psi_4(x)\end{pmatrix} = D(g)\begin{pmatrix}\psi_1(T(g)^{-1}x)\\ \vdots\\ \psi_4(T(g)^{-1}x)\end{pmatrix}$$

with $D(g)$ some 4-dim. repre of G

- Hamiltonian transformation

$$H \overset{g}{\longmapsto} \mathcal{U}(g) H \mathcal{U}(g)^+$$

- $g \in G$ symmetry group of a system $\Leftrightarrow$ H invariant

$$\Rightarrow H\,\mathcal{U}(g) = \mathcal{U}(g) H \quad \forall g \in G$$

Note: • $\mathcal{U}(g)$ is not from IRREP $\Rightarrow$ does not imply $H = \lambda \mathbb{1}$
but ...

- eigenfunctions

$$H\psi = \lambda \psi \overset{g}{\longmapsto} \mathcal{U}(g) H \psi \overset{[H,\mathcal{U}]=0}{=} H\,\mathcal{U}(g)\psi = \lambda\,\mathcal{U}(g)\psi$$

$\Rightarrow$ subspace $\mathcal{H}_\lambda \subset \mathcal{H}$ of eigenfunctions corresp. to (possibly degenerate) λ is invariant under action of G

$\Rightarrow$ basis of $\mathcal{H}_\lambda$ forms basis of a repre of G on $\mathcal{H}_\lambda$:

$$\mathcal{U}(g)\psi_{\lambda,n} = \sum_m \psi_{\lambda,m} D^\lambda(g)^m{}_n \qquad \mathrm{span}\left(\{\psi_{\lambda,1}, \dots, \psi_{\lambda,d}\}\right) = \mathcal{H}_\lambda$$

1, $\mathcal{H}_\lambda$ does not contain proper invar. subspace

$\Rightarrow$ D^λ is IRREP & it's dimension corresponds to the degree of degeneracy of λ

(note that indeed $H = \lambda\mathbb{1}$ on $\mathcal{H}^\lambda$ as required by Schur)

$\Rightarrow$ symmetry explains degeneracy of an energy level:

- if the eigenfunc. transforms as multi-dim IRREP then the level must be degenerate
- ground state typically totally sym $\leftrightarrow$ trivial IRREP
  $\Rightarrow$ non-degenerate

$\Rightarrow$ this is normal/geometrical degeneracy

2, $\mathcal{H}_\lambda$ reducible

a) accidental degeneracy

- due to specific values of some constants (ie, for specific geometry)

b) hidden (usually dynamical) symmetry

- true sym. group is larger $\Rightarrow$ higher-dim. IRREPs

Examples: 1, hidden symmetry – H atom

- apparent sym. group is $SO(3)$ $\Rightarrow$ $E_{nl} = E_n$ seems accidental
- additional symmetry – Laplace-Runge-Lenz vector

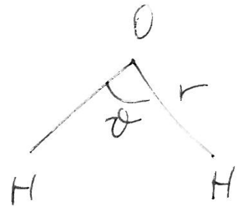
(remember Kepler problem: $\vec{A} = \vec{p} \times \vec{l} - mk\hat{r}$)

$$A_q = -mk\hat{r}_q + \frac{1}{2}\varepsilon_{qij}(p_i l_j + l_j p_i) \quad \Leftarrow [l_i, p_j] \neq 0$$

$\Rightarrow$ full sym. group is $SO(4)/\mathbb{Z}_2 \sim SO(3) \otimes SO(3)$

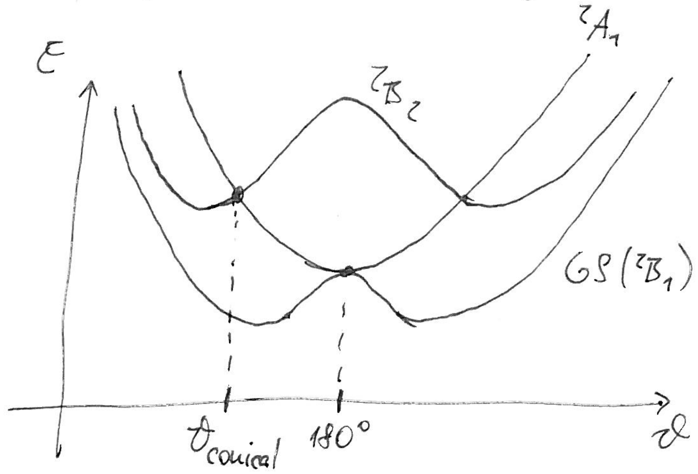
$\Rightarrow$ explains the degeneracy

2, accidental degeneracy – metastable states of H_2O^-



$\Rightarrow$

C_{2v}	E	C_2	σ_v	σ_v'
A_1	1	1	1	1
A_2	1	1	-1	-1
B_1	1	-1	1	-1
B_2	1	-1	-1	1



- $\vartheta_{conical}$ ... true accidental degeneracy (G is C_{2v})
- $\vartheta = 180°$ ... sym. group is $D_{\infty h}$ & GS is of Π_u sym.

$C_{2v} \leftrightarrow D_{\infty h}$:

	$E \leftrightarrow E$	$C_2 \leftrightarrow \infty C_2$	$\sigma_v \leftrightarrow S_1$	$\sigma_v' \leftrightarrow \infty\sigma_v$	
Π_u	2	0	2	0	$= A_1 \oplus B_2$

NOTE: • here we consider $C_{2v} < D_{\infty h}$ & this is subduction!

$D_{\infty h}$ point group

not Abelian, ∞ irreducible representations

Character table

	E	$2C_\infty^\varphi$	...	$\infty\sigma_v$	i	$2S_\infty^\varphi$	...	$\infty C'_2$	linear functions, rotations	quadratic
$A_{1g}=\Sigma^+_g$	1	1	...	1	1	1	...	1		x^2+y^2, z^2
$A_{2g}=\Sigma^-_g$	1	1	...	-1	1	1	...	-1	R_z	
$E_{1g}=\Pi_g$	2	2cos(φ)	...	0	2	-2cos(φ)	...	0	(R_x, R_y)	(xz, yz)
$E_{2g}=\Delta_g$	2	2cos(2φ)	...	0	2	2cos(2φ)	...	0		(x^2-y^2, xy)
$E_{3g}=\Phi_g$	2	2cos(3φ)	...	0	2	-2cos(3φ)	...	0		
...	...	...	...	...	...	...	...	...		
$A_{1u}=\Sigma^+_u$	1	1	...	1	-1	-1	...	-1	z	
$A_{2u}=\Sigma^-_u$	1	1	...	-1	-1	-1	...	1		
$E_{1u}=\Pi_u$	2	2cos(φ)	...	0	-2	2cos(φ)	...	0	(x, y)	
$E_{2u}=\Delta_u$	2	2cos(2φ)	...	0	-2	-2cos(2φ)	...	0		
$E_{3u}=\Phi_u$	2	2cos(3φ)	...	0	-2	2cos(3φ)	...	0		
...	...	...	...	...	...	...	...	...		