Lecture synopsis

NTMF061: Group theory and its application in physics

Winter term 2020/21

Week 1: October 2^{nd}

- **group definition, order of the group**, examples of the groups, Abelian group
- multiplication table, **rearrangement theorem:** *Each row and each column of the multiplication table contains each element of the group once and only once.*
- **subgroup**, order of an element, cyclic subgroup, **theorem:** *Intersection of two subgouprs of G is again a subgroup of G.*
- **left and right cosets with respect to a subgroup**, *each element of G is a member of one and only one left/right coset with respect to a given subgroup*
- **Lagrange theorem:** *Order of a subgroup of G divides $\#G$*, index of a subgroup

Tutorial: Classification of point groups, symmetry elements and symmetry operations

Week 2: October 9^{th}

- **conjugacy classes**, **theorem:** *Number of elements in any class (g) is a divisor of $\#G$.*
- **normal (invariant) subgroup**, center of a group, simple and semi-simple groups
- **theorem:** $H \triangleleft G \Leftrightarrow H$ *consists entirely of complete classes of G.*
- product of left/right cosets, **theorem (factor group):** *The set of all distinct cosets with respect to an invariant subgroup $H \triangleleft G$ forms a factor (quotient) group.*
- **homomoprhic mapping**, kernel and image of a homomorphism
- surjective, injective and bijective (**isomorphic**) mappings
- **theorem:** *Let $\Phi : G \to G'$ be a homomorphism. Then* $\mathrm{Im}\,\phi$ *is a subgroup of G',* $\mathrm{Ker}\,\phi$ *is invariant subgroup of G and* $Im\,\phi \sim G/\mathrm{Ker}\,\phi$.
- canonical projection of G onto G/H: $g \mapsto gH$
- **direct and semi-direct product groups**, Euler group as a semi-direct group

<u>Week 3: October 16</u>

- (left/right) **group action on a set**, orbit, stabilizer (isotropy) group, **theorem:** *Let G be a finite group acting on a set $\mathcal{M}$. Then $(\#G \cdot m)(\#G_m) = \#G$.*
- Group action on itself: left/right translation, conjugation
- **representation of a group as an action on a vector space** (homomorphism to the group of all automorphisms on the vector space), dimension of the representation, faithful representation, **matrix representation**
- **equivalent representations**, intertwining mapping
- equivalent matrix representations are related by similarity transformation
- invariant subspace under group action, **reducible and irreducible representation**, subrepresentation, reducibility of matrix representations
- **completely reducible representation**, block-diagonal form of completely reducible matrix representation
- **theorem:** *Every irreducible representation of a finite group is finite-dimensional.*

<u>Week 4: October 23</u>

- **unitary representation**, **theorem:** *Every finite-dimensional reducible unitary representation of a group G is completely reducible.*
- **theorem:** *Every finite-dimensional representation of a finite or compact Lie group is equivalent to some unitary representation.*
- **theorem (Maschke):** *Every finite-dimensional reducible representation of a finite or compact Lie group is completely reducible.*
- **Schur lemma I:** *Intertwining mapping between two irreducible representations is either isomorphic (and the two representations are equivalent) or null mapping.*
- **Schur lemma II:** *Let (ρ, V) be a complex finite-dimensional irreducible representation of a group G and S an intertwining operator on V commuting with all operators $T(g)$ $\forall g \in G$ from the representation. Then $S = \lambda \mathbb{1}$ for $\lambda \in \mathbb{C}$.*
- **theorem:** *Complex finite-dimensional irreducible representations of an Abelian group are one-dimensional.*
- **theorem:** *Orthogonality relations for irreducible matrix representations*

$$\sum_{g \in G} [D^{\mu}(g)_i^j]^* D^{\nu}(g)_l^k = \frac{\#G}{d_\mu} \delta_{\mu\nu} \delta_{jk} \delta_{il}$$

- **character of a representation**
- **theorem:** *Orthogonality relations for characters*

$$\sum_{g \in G} \chi^\mu(g)^* \chi^\nu(g) = \#G \delta_{\mu\nu}$$

- **theorem:** *For finite or compact Lie group, equality of characters of two representations is sufficient condition for their equivalence.*
- decomposition of a reducible representation ρ (of a finite or compact Lie group):

$$\rho = \oplus_\mu n_\mu \rho^\mu \Longrightarrow n_\mu = \frac{1}{\#G} \sum_g \chi^\mu(g)^* \chi(g)$$

  with summation running over all non-equivalent IRREPs ρ^μ.
- regular representation of a finite group, **theorem:** $\#G = \sum_\mu d_\mu^2$

Week 5: October 30

- multiplication of conjugacy classes, class constants $(g_i)(g_j) = \sum_{(g_k)} c_{ij}^k (g_k)$
- **theorem:** *Number of non-equivalent IRREPs of a finite group is equal to the number of distinct conjugacy classes.*
- **theorem (Frobenius):** *Representation (ρ, V) of a finite group G is irreducible $\Leftrightarrow \sum_{(g_k)} n_k \chi(g_k)^* \chi(g_k) = \#G$.*

Tutorial:

1. Vector and pseudo-vector representation of $O(3)$
2. Character table for D_{3h}

Week 6: November 6

- **direct product representation**, symmetric and anti-symmetric products of equivalent representations
- **symmetrization (projection) operators** (complete and incomplete), construction of a basis of irreducible (sub)representation

Tutorial

1. Character table for D_{3h} part II (transformation of quadratic functions)
2. basis of s orbitals for the H_3^{2+} molecule

Week 7: November 15

- relations between representations of a group and its subgroups
  - **subduced and induced representations**, decomposition to irreducible representations
  - **theorem (Frobenius reciprocity):** $\alpha_\mu^{\nu\uparrow G} = \alpha_\nu^{\mu\downarrow H}$
- **symmetries in quanum mechanics**
  - transformation of a wave function (group action on a Hilbert space $\mathcal{L}^2(\mathbb{R}^3)$
  - transformation of an operator
  - symmetry group as a group of transformations leaving invariant the Hamiltonian of a system
  - eigenfunctions of the Hamiltonian as bases of irreducible representations of the symmetry group, degeneration of energy levels (normal and accidental, hidden symmetries)

Tutorial

1. Induced representations and Frobenius reciprocity
   Subduced representations in quantum mechanics: splitting of atomic energy levels in a crystal latice

Week 8: November 22

- decomposition of direct product representation – **Clebsh-Gordan series**
- basis of direct product representation – **Clebsh-Gordan coefficients**
- selection rules for **matrix elements of invariant scalar operators**
- **irreducible tensor operators**, **Wigner-Eckart theorem**
- molecular vibrations and optical transitions
  - **normal coordinates** (vibrational modes) as bases of IRREPs of the symmetry group
  - activity of vibrational modes in infrared spectrum and in Raman scattering

Tutorial

1. MO-LCAO for H_3^{2+} – the Hamiltonian
2. Optical transitions in CO_3^{2+} ion
3. Normal coordinates for diatomic molecule

Week 9: November 27

LIE GROUPS

- **$SO(3)$ as a group of orthogonal matrices 3×3 with unit determinant**
  - **linearization** – antisymmetic matrices as generators of infinitezimal rotations
  - general rotation as **exponential** of the generators
  - group $O(3)$ has the same generators but exponential mapping covers only the connected subgroup $SO(3)$
  - generators of rotations form **Lie algebra $\mathfrak{so}(3)$** with **structure constants**

$$[J_i, J_j] = ic_{ij}^k J_k, \qquad c_{ij}^k = \varepsilon_{ijk}$$

  - $(J_i)_{jk} = -ic_{ij}^k$ is **adjoing representation** of the Lie algebra $\mathfrak{so}(3)$

- **Review of differential geommetry**
  - **topological space**, open and closed sets, neighborhood of a point, **continuous mapping**, homeomorphism
  - **connected**, **path-connected** and **simply-connected** topological spaces, **compactness**
  - **topological manifold**, **coordinate map**, atlas, **differentiable manifolds** (smooth, analytical)

- **Lie groups as smooth manifolds**
  - topological group, **smooth mapping**, **real Lie group**, linear group
  - global topological properties of Lie groups – $E(2)$, $SO(2)$, $SO(3)$, $SU(2)$

Week 10: December 4

LIE ALGEBRAS – left-invariant vector fields on Lie groups

- **tangent vectors** as a class of equivalence of tangent curves, tangent space T_pM, directional derivative, **isomorphism of T_pM and the space of derivatives** D_pM, tangent bundle
- **vector field**, **integral curve** of a vect. field
- **push-forward mapping**
- **Lie bracket**
- **left-invariant vector field**, **isomorphism of T_eG and the space** $\mathcal{L}(G)$ of left-invariant fields on a Lie group G

- push-forward of Lie bracket, commutator of vectors from T_eG using Lie bracket of the corresponding fields from $\mathcal{L}(G)$, T_eG **as Lie algebra of** G
- theorem (Ado): Every finite-dimensional abstract LA is isomorphic to some LA of square matrices with std. commutator.

Exponential mapping

- **one-parameter subgroup** of a LG
- **theorem:** Every one-parameter subgroup of G is integral curve of some left-invariant vector field and every integral curve of a left-invariant vector field is one-parameter subgroup.
- **theorem:** Left-invariant vector fields on G are complete.
- **exponential mapping** from LA $\mathcal{G}$ to LG G:

  $$\exp : \mathcal{G} \to G \quad X \mapsto \exp(X) \equiv \gamma^X(1)$$

  for $\gamma^X(t) \subset G$ the one-parameter subgroup corresponding to $X \in \mathcal{G}$.
- $\gamma^X(t) = \exp(tX)$

Tutorial:

1. **matrix groups** and their algebras (left-invariant fields, structure constants and commutator on T_eG) – $\mathfrak{gl}(n, \mathbb{R})$

<u>Week 12: December 11</u>

Exponential mapping

- **theorem:** Exponential mapping is local diffeomorphism between T_eG and $U(e) \subset G$
- connected subgroup, **theorem:** Let G be compact LG, then every element of its connected subgroup can be written as $g = \exp(X)$ for some $X \in \mathcal{G}$.
- **theorem:** Every connected component of a LG is a right coset with respect to connected subgroup.
- **theorem:** Every point from the connected subgroup of G can be written as a finite product of exponential elements.

Relations between Lie groups and their Lie algebras

- **homomoprhism and isomorphism** between LAs
- derived homomoprhism of LAs, LA of a subgroup of G is subalgebra of the LA of G
- **theorem:** "Let Φ be isomorphism between two LGs. Then derived homomoprhism Φ_* is isomorphism between corresponding LAs."
- discrete subgroup, **theorem:** "If the kernel of a surjective homomoprhism Φ between two LGs is discrete, then the derived homomoprhism Φ_* is isomorphism between corresponding LAs."
- relation between non-isomorphic LGs with isomorphic LAs, **universal covering group**

Tutorial:

1. Homomorphism (double covering) $SL(2,\mathbb{C}) \to L_+^\uparrow$ and $SU(2) \to SO(3)$

<u>Week 13: 18.12.</u>

Killing-Cartan form – recovering the geometry of LG from LA

- invariant subalgebra, **simple** and **semi-simple LA a LG**
- representation of LA on V as a homomoprhism $\mathcal{G} \to \mathrm{End(V)}$, **matrix representation of LA**
- **adjoint representation** of LA
- **Killing-Cartan form** and metric, properties of K-C form
- solvable LA, **theorem:** Cartan criteria for solvable and **semi-simple** LA
- **theorem:** LG G is compact if and only if the C-K form on corresponding LA is negative definite.

Representations of LA

- analytical representation of a LG, **theorem:** "Relations between analytical representation of LG and corresponding representation of LA"
- **theorem:** "Relations between reducibility of representations of LG and LA"
- relations of representations of $SO(3)$ and $\mathfrak{so}(3) \sim \mathfrak{su}(2)$, multivalued representations and the universal covering group
- **complexification of LA**, relations of representations of real and complexified LAs

Tutorial:

1. Killing-Cartan form on $\mathfrak{sl}(2,\mathbb{R})$