TOV equations for structure of spherical star

In Misner-Thorne-Wheeler texbook we have the following equations

$$\frac{dp}{dr} = -\frac{(\rho+p)(m+4\pi r^3 p)}{r(r-2m)}$$
$$\frac{dm}{dr} = 4\pi r^2 \rho$$
$$\frac{d\phi}{dr} = \frac{m+4\pi r^3 p}{r(r-2m)}$$

Here $\rho = \rho_0 + \epsilon$ where $\rho_0 = n_{\rm b} m_{\rm b}$ is rest-mass density and ϵ internal energy. We assume polytropic equation of state (see eqs. (1.74) and (1.86) in [Baumgarte&Shapiro])

$$p = k\rho_0^\gamma,$$

i.e.

$$\rho_0 = (p/k)^{1/\gamma} \quad \text{and} \quad \rho = \rho_0 + \frac{p}{\gamma - 1}$$

We put $k = 1$ and consider only dimensionless problem (see [Baumgarte&Shapiro]for scaling of true quantities such as stellar radius or mass for $k \neq 1$).

We get only inner part of the metric

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1 - \frac{2m(r)}{r}} + r^2 d\Omega^2$$

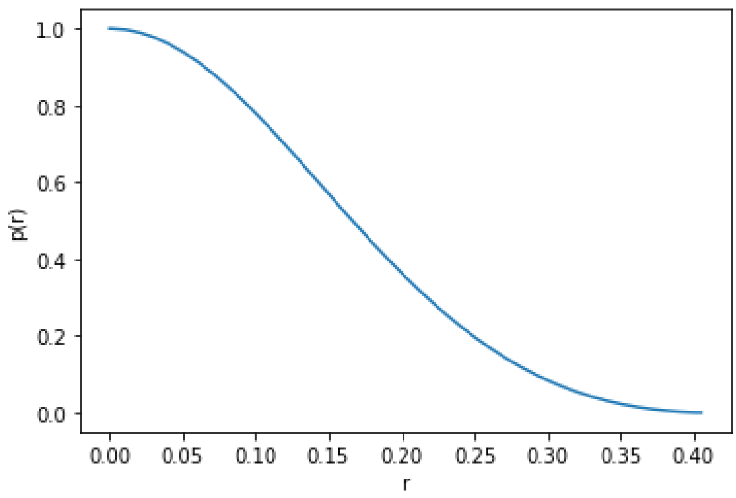
Note that the code does not solve function $\Phi(r)$ completely, because in the first step we simply assume $\Phi(r=0) = 0$ which is not OK, the right boundary condition is $\Phi(\infty) = 0$ or $e^{2\phi(r_s)} = 1 - 2m(r_s)/r_s$ at surface of the star.

Problem: Find the coordinate time a null particle (neutrino) needs to get from the center of the star to its surface. (*Hint*: compute the integral as solution of ODE.)

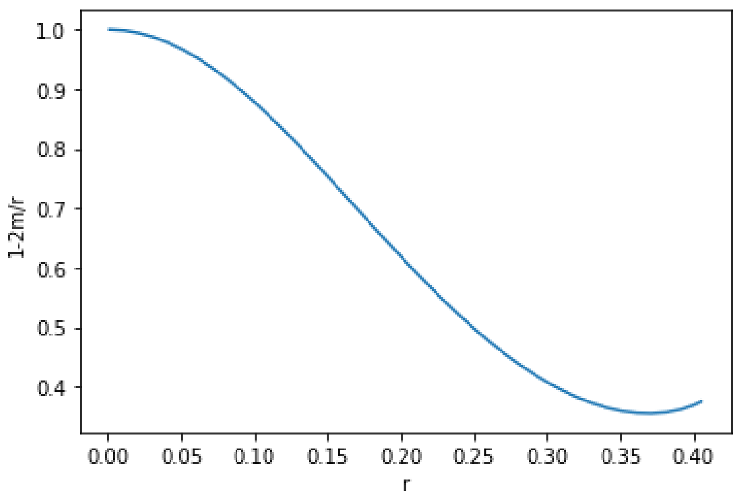
```
import numpy as np
from scipy.integrate import solve_ivp
import matplotlib.pyplot as plt
```

```
# EOS polytropic index
gamma = 2.7
Pi    = np.pi

# ODE solver output
dr_Max = 0.01

# central pressure
p0    = 1

# indices of functions p(r), m(r), phi(r)
ip, im, iphi = (0,1,2)
```

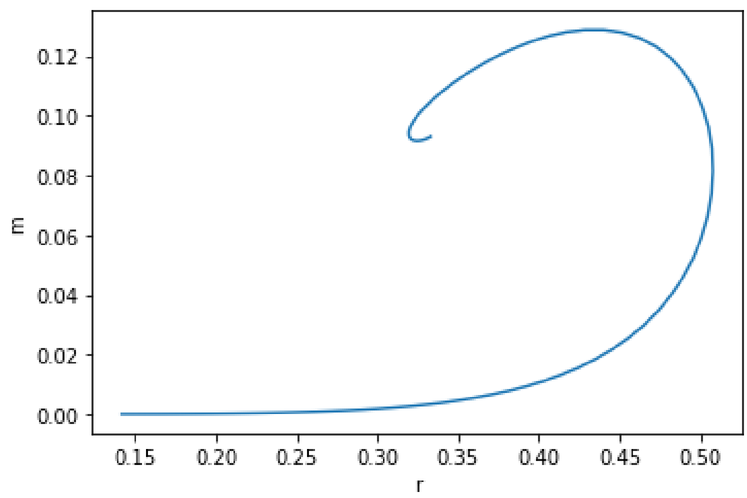
```
# equations
def TOVeq(r, U):
    p, m, phi = U

    rho = 0
    if p>0:
        rho = p**(1/gamma)+p/(gamma-1)

    dphidr = 0
    if r>0:
        dphidr = (m + 4*Pi * r**3 * p)/(r*(r - 2*m))

    dpdr = -(rho + p)*dphidr
    dmdr = 4*Pi * r**2 * rho

    return [dpdr, dmdr, dphidr]


# Terminate integration at the stellar surface
def zero_p(r, U):
    p, m, phi = U
    return p

zero_p.terminal = True

# initital/boundary values
# note that phi(infty)=0 not phi(0), so the result of integration phi(r) has to shift
U0 = [p0,0,0]

sol =  solve_ivp(TOVeq, (0,10), U0, events=zero_p, max_step=dr_Max)

plt.plot(sol.t, sol.y[ip])
plt.xlabel("r")
plt.ylabel("p(r)")
plt.show()

plt.plot(sol.t[2::], 1-2*sol.y[im,2::]/sol.t[2::])
plt.xlabel("r")
plt.ylabel("1-2m/r")
plt.show()

#print( sol.t[-1], sol.y[::,-1])

def rMaxM(p0):
    U0 = [p0,0,0]
    sol =  solve_ivp(TOVeq, (0,4), U0, events=zero_p, rtol=1E-6, atol=1E-8)
    return  [ sol.t[-1] , sol.y[im,-1] ]

P_list = np.exp( np.arange(-15,6,0.25) )
mr = np.array([ rMaxM(p) for p in P_list ])
plt.plot(mr[::,0],mr[::,1])
plt.xlabel("r")
plt.ylabel("m")
plt.show()
```



1.0
0.8
0.6
0.4
0.2
0.0
p(r)
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0.35
0.40
r



1.0
0.9
0.8
0.7
0.6
0.5
0.4
1-2m/r
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0.35
0.40
r



0.12
0.10
0.08
0.06
0.04
0.02
0.00
m
0.15
0.20
0.25
0.30
0.35
0.40
0.45
0.50
r