Cosmological models and stability

Lars Andersson

I would already have concluded my researches about world harmony, had not Tycho's astronomy so shackled me that I nearly went out of my mind. *Johannes Kepler*

Letter to Herwart, quoted in [1, p. 127]

Abstract Principles in the form of heuristic guidelines or generally accepted dogma play an important role in the development of physical theories. In particular, philosophical considerations and principles figure prominently in the work of Albert Einstein. As mentioned in the talk by Jiří Bičák at this conference, Einstein formulated the equivalence principle, an essential step on the road to general relativity, during his time in Prague 1911-1912. In this talk, I would like to discuss some aspects of cosmological models. As cosmology is an area of physics where "principles" such as the "cosmological principle" or the "Copernican principle" play a prominent role in motivating the class of models which form part of the current standard model, I will start by comparing the role of the equivalence principle to that of the principles used in cosmology. I will then briefly describe the standard model of cosmology to give a perspective on some mathematical problems and conjectures on cosmological models, which are discussed in the later part of this paper.

1 Introduction

As stated by Einstein in his paper from 1912 [2], submitted just before his departure from Prague, the equivalence principle is *"eine Natürliche Extrapolation einer der allgemeinsten Erfahrungssätze der Physik"*[1], and can consequently be claimed to be exactly valid on all scales. Since the equivalence principle is compatible with Einstein's relativity principle of 1905 only in the limit of constant gravitational potential, accepting the principle of equivalence meant that a new foundation for the theory of gravitation must be sought. The challenge of doing so, which Einstein in

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[1] "a natural extrapolation of one of the most general empirical propositions of physics"

his 1912 paper poses to his colleagues: *"Ich möchte alle Fachgenossen bitten, sich an diesem wichtigen Problem zu versuchen!"*, is one that he himself devoted the coming years to, finally arriving at the 1915 theory of general relativity.

General Relativity describes the universe as a 4-manifold $\mathbf{M}$ with a metric $\mathbf{g}_{\alpha\beta}$ of Lorentzian signature. The Einstein equations,

$$\mathbf{R}_{\alpha\beta} - \frac{1}{2}\mathbf{R}\mathbf{g}_{\alpha\beta} + \Lambda\mathbf{g}_{\alpha\beta} = 8\pi G\mathbf{T}_{\alpha\beta}\,, \tag{1}$$

originally given in [3], relate the geometry of spacetime $(\mathbf{M}, \mathbf{g}_{\alpha\beta})$ to matter fields with energy-momentum tensor $\mathbf{T}_{\alpha\beta}$. By the correspondence principle, the stress energy tensor $\mathbf{T}_{\alpha\beta}$ should correspond to the stress energy tensor of a special relativistic matter model, and in particular be divergence free. For "ordinary matter" one expects $\mathbf{T}_{\alpha\beta}$ to satisfy energy conditions such as the dominant energy conditon. Here I have included the "cosmological constant term" $\Lambda\mathbf{g}_{\alpha\beta}$ in (1), which was not present in the equations given in [3]. The left hand side of (1), where $\mathbf{R}_{\alpha\beta}$ is the Ricci tensor, $\mathbf{R}$ is the Ricci scalar and Λ is a constant, is the most general covariant tensor expression of vanishing divergence, depending on $\mathbf{g}_{\alpha\beta}$ and its derivatives up to second order, and linear in second derivatives. Further, its left hand side is the most general second order Euler-Lagrange equation, derived by varying a covariant Lagrange density defined in $\mathbf{g}_{\alpha\beta}$ and its first two derivatives, see [4, 5] and references therein. The covariance of the equations of general relativity under spacetime diffeomorphisms, makes the theory compatible with the strong version of the equivalence principle.

Since it can be claimed to be exactly valid, the equivalence principle is subject to empirical tests and there is a long history of experiments testing various versions of the (weak or strong) equivalence principles, see e.g. [6], see also [7] in this volume. Until the present, the equivalence principle has survived all experimental tests, and an experiment clearly demonstrating a deviation from the predictions based on the equivalence principle would necessitate a revision of the foundations of modern physics.

The arguments of the physicist and philosopher Ernst Mach played an important role in the development of Einstein's ideas leading up to general relativity, including the formulation of the equivalence principle. The fact that in general relativity, matter influences the motion of test particles via its effect on spacetime curvature means that in contrast to Newtonian gravity, the "action at a distance" which was critizised by Mach is not present in general relativity, which hence agrees with the guiding idea which Einstein referred to as "Mach's principle", i.e. loosely speaking the idea that the distribution of matter in the universe determines local frames of inertia, see [8], see also [9]. The role of Mach's principle in the context of cosmology is discussed in [10]. This played a central role in Einstein's development of general relativity, and also in his discussion of general relativistic cosmology, but it appears difficult to formulate experimentally testable consequences, cf. [11], although Mach's principle has of course been brought up in connection with "Newton's bucket" and frame dragging. The book [12] gives an excellent overview of issues related to Mach's principle. However, the principles which are most relevant

for the present discussion are the hierarchy of "cosmological principles", for example the cosmological principle of Einstein and the perfect cosmological principle of Bondi, Gold and Hoyle. See [13, §2.1] for an overview of the cosmological principles. These principles play a role which is fundamentally different from that of the equivalence principle, in the sense that they do not make predictions which are expected to be exactly true at all scales. At best, they can be viewed as simplifying assumptions that enable one to construct testable physical models.

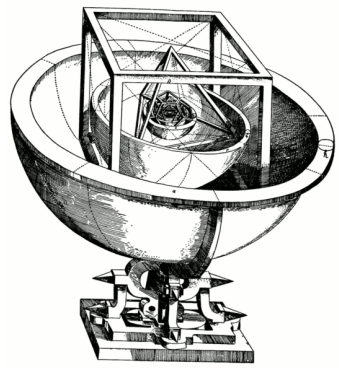


Fig. 1 Kepler's model of the solar system based on Platonic solids, from Mysterium Cosmographicum (1596).

The work of Kepler, who is perhaps more intimately connected with Prague than Einstein, provides an interesting illustration of the relationship between theoretical principle and observation. In the time of Kepler, the world-model of Copernicus had placed the sun at the center of the universe and described the planets as moving on circular orbits around it. Not long before his move to Prague in 1600, Kepler believed himself to have completed the Copernican world-model based on the mathematical perfection of circles, by adding to it an element of equal perfection and beauty, namely the geometry of the Platonic solids, which according to Kepler's expectations would determine the sizes of the planetary orbits.

Fortunately, it was possible for Kepler to use Tycho Brahe's observational data to test the predictions of his model. To his deep consternation Kepler realized that the planets do not, after all, move on circular orbits. The beautiful principles which had inspired Kepler to laboriously analyze the observational data of Tycho had to be discarded. In analyzing the data, Kepler not only discovered his three laws of planetary motion but also came close to introducing the notion of force which became fully clear only through the work of Newton. One could say that through the work of Kepler and later Newton, one set of "a priori" principles (those of Copernicus and Kepler) were replaced by a model based on the dynamical laws of Newtonian gravity.

1.1 The Cosmological principle

Although Newtonian ideas continued to dominate physics throughout the 19th century, there were well known anomalies of a theoretical as well as observational nature, and these served as a guide for the developments of the early 20th century. The conflict between the covariance of Maxwell theory under the Lorentz group and the more restricted invariance properties of the Newtonian laws led to the introduction of special relativity. Similarly, as discussed above, the incompatibility of special relativity and gravitation led to the development of general relativity. The explanation of the anomalous precession of the perihelion of Mercury [14][2] by general relativity was, together with its new prediction for the deflection of light by the sun, confirmed by subsequent observations [17], were among the factors which led to its rapid acceptance.

Among the main paradoxes of Newtonian physics and world view in applications to cosmology were Olbers' paradox and the incompatibility of Newtonian gravity with infinitely extended homogenous matter distributions, which had prevented the construction of a cosmological model consistent with Newtonian ideas. This latter fact, which had been elucidated by von Seeliger and others, see [18] for discussion and references, played an important role in Einstein's reasoning about cosmological models in his 1917 paper [19], in particular in motivating the introduction of the cosmological constant in that paper.

As has already been mentioned, the philosophy of Mach, albeit firmly based in Newtonian physics, was an important source of inspiration for Einstein. However, incorporating Machian ideas in a general relativistic cosmology presented serious difficulties. After some early attempts had been discarded, Einstein in [19] adopted a spatially homogenous model of the universe as a means of making a general relativistic cosmology compatible with Machian ideas. Introducing a "cosmological constant" term $\Lambda \mathbf{g}_{\alpha\beta}$ in the field equation of general relativity, which Einstein first motivated through a discussion of homogenous matter distributions in Newtonian gravity, and assuming that there is a family of observers who see the same matter density everywhere, led to a static universe filled with a homogenous and isotropic matter distribution. The spacetime of the Einstein model is a Lorentzian cylinder. The line element takes, up to a rescaling, the form

$$ds^2 = \mathbf{g}_{\alpha\beta} dx^\alpha dx^\beta = -dt^2 + g_{S^3} \, .$$

This give a solution to (1) with positive Λ, and with matter consisting of a pressureless fluid with everywhere constant energy density.

Shortly after Einstein's initial work on a static general relativistic cosmology, Friedmann [20] proposed a model of an expanding universe

$$ds^2 = -dt^2 + a^2(t) g_\kappa \tag{2}$$

[2] From the perspective of the current situation in physics, it is amusing to recall that attempts had been made in the 19th century to explain the obseved precession of Mercury both by dark matter [15] (the planet Vulcan hypothesis) as well as modifications of gravity [16].

where $a(t)$ is a scale factor, g_κ for $\kappa = +1, 0, -1$ is the sperical, flat or hyperbolic metric. Line elements of the form (2) are also called Robertson-Walker line elements, see below. During the 1920s, Lemaître and Hubble showed, based on observational work of Slipher, Humason and others, that redshift increases with distance leading to the Hubble law, see figure 2, which fits with the expanding Friedmann models. In the context of the expanding Friedmann models, Olbers' paradox can

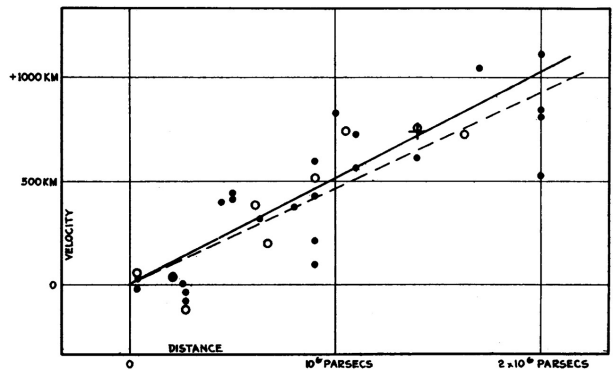


Fig. 2 Hubble's original 1929 graph [21].

be resolved. Expanding Friedmann models containing ordinary matter have $a \searrow 0$ at some time in the past, where spacetime curvature and matter densities diverge. These models led, via the work of Lemaître, Gamow, Hoyle and others, to the hot big bang model which is the basis for the cosmological models in use today.

E. A. Milne criticized the big bang models on the basis that they introduced an extraneous "cosmic time" and also that they lacked explanatory power (e.g. the sign of the spatial curvature is a priori undetermined). Instead, he proposed an extension of what he termed "Einstein's cosmological principle", to the effect that "The universe must appear the same to all observers" [22]. Milne added to this the postulate that observations are interpreted by each observer according to the principles of special relativity and argued that this "extended relativity principle" led to an essentially unique cosmological model.

The derivation of the general form of the line element compatible with the isotropy of the universe, and also with Einstein's cosmological principle in the sense discussed by Milne was given by Robertson [23] and Walker [24] around the same time, and found to be of the same form as that used by Friedmann and Lemaître in their cosmological models. As pointed out by Robertson [25], the general relativistic line element compatible with Milne's cosmology is a special case of (2), namely the empty $\kappa = -1$ universe, which is locally isometric to Minkowski space. This is therefore known as the Milne model.

It was a similar dissatisfaction with the lack of predictivity of general relativistic cosmology that led Bondi, Gold and Hoyle [26, 27] to introduce the "perfect cosmological principle", which is essentially a version of the postulate of Milne, but

viewed from the perspective of general relativity. By allowing for creation of matter, they showed that it is possible to construct an expanding cosmological model satisfying this principle. However, the perfect cosmological principle tightly constrains the possible models of the universe and the resulting steady state model is considered to be incompatible with observations. The book of Kragh [28] contains an interesting discussion of the conflict between the steady state model and the now-standard "big bang" cosmology.

From the current perspective, it may be said that the introduction of what Milne called Einstein's cosmological principle led to a class of general relativistic cosmological models. By introducing a collection of perfect fluids, a much simplified version of the problem of cosmological modelling reduces to the problem of fitting a relatively small number of parameters to observational data, which could be said to put cosmology on a similar footing as high energy particle physics. Indeed, as mentioned by Peebles [29, Chapter I], it was Weinberg [30] who introduced the notion, borrowed from high energy particle physics, of a "standard model" into cosmology.

At present, with the tremendous influx of data from observations of many different types and at many different wavelengths, including observations of the cosmic microwave background and galaxy surveys, it is often stated that we are entering an era of precision cosmology. However, the widening range of observational methods makes the process from observations to parameter estimation increasingly complex. In particular, the prominent role of simplifying assumptions or principles in the formulation of cosmological models and the model depence in the analysis of astronomical data, makes it important to keep in mind the difference between a model which fits data to a high degree of precision and a model which accurately describes the actual universe [31].

2 Cosmological models

For a Friedmann model, with line element of the form (2), the stress energy tensor has the form

$$\mathbf{T}_{\alpha\beta} = \rho \mathbf{u}_\alpha \mathbf{u}_\beta + p(\mathbf{g}_{\alpha\beta} + \mathbf{u}_\alpha \mathbf{u}_\beta)\,,$$

which is compatible with perfect fluid matter. Here u^α is the unit timelike normal to the t level sets, which in the special case of the Friemann model coincides with the normalized 4-velocity of the fluid particles, ρ is the energy density of the matter and p is the pressure. We consider matter and radiation as described by a collection of fluids, indexed by i, with linear equations of state,

$$p_i = \omega_i \rho_i.$$

The Hubble constant (i.e. up to a constant factor the mean curvature of the t level sets) is

$$H = \dot{a}/a$$

In the special case of a Friedmann model, the contribution of the curvature of the t level sets in the Einstein equations can be described in terms of a fluid with equation of state $p = -\rho/3$, while the effect of the cosmological constant can be described by a fluid satisfying $p = -\rho$. Thus if we consider a simple model containing a fluid with pressure zero (dust), and with a cosmological constant Λ, this can be described by introducing the dimensionless density parameters

$$\begin{aligned} \Omega_m &= \frac{8\pi}{3H^2}\rho_m, && \text{“Matter”: } \omega = 0, \\ \Omega_\kappa &= -\frac{\kappa}{a^2H^2}, && \text{“Curvature”: } \omega = -1/3, \\ \Omega_\Lambda &= \frac{8\pi}{3H^2}\rho_\Lambda, && \text{“Vacuum”: } \omega = -1. \end{aligned}$$

The model can be parametrized by the present values

$$\Omega_{m0}, \Omega_{\kappa 0}, \Omega_{\Lambda 0},$$

of the density parameters. The conservation of matter and equation of state implies that the fluid densities ρ_i depend only on the scale factor

$$\rho_i \propto a^{-3(1+\omega_i)}. \tag{3}$$

The Hamiltonian constraint (i.e. the projection of the Einstein equations (1) on u^α) takes the form

$$\Omega_m + \Omega_\kappa + \Omega_\Lambda = 1, \tag{4}$$

which, using (3), can be written as

$$\frac{H^2}{H_0^2} = \Omega_{0m}\left(\frac{a_0}{a}\right)^3 + \Omega_{0\Lambda} + \Omega_{0\kappa}\left(\frac{a_0}{a}\right)^2. \tag{5}$$

Here H_0, a_0 are the present value of the Hubble constant and of the scale factor respectively. Due to the uncertainty in the value of H_0, it is usually given in terms of a dimensionless parameter h as

$$H_0 = 100\,h\text{km s}^{-1}\text{ Mpc}^{-1}.$$

Equation (5) can be integrated to relate observable quantities, e.g. redshift and luminosity distance, for given values of the parameters $H_0, \Omega_{m0}, \Omega_{\kappa 0}, \Omega_{\Lambda 0}$.

It is convenient to study the global behavior of Friedmann models in terms of the dimensionless density parameters. This analysis is explained in [32, Chapter 2], see also [33, 34]. Due to the Hamiltonian constraint (4), we have $\Omega_\kappa = 1 - \Omega_m - \Omega_\Lambda$.

The fixed points of the dynamial system in the $(\Omega_m, \Omega_\Lambda)$ plane are the Einstein-de Sitter big-bang model $\mathbf{Bb} = (1,0)$ and the spatially flat de Sitter model $\mathbf{dS} = (0,1)$, as well as the empty $\kappa = -1$ Milne model $\mathbf{M} = (0,0)$. One finds that **Bb** is a source and **dS** is a sink, while **M** is a saddle point. The static Einstein universe has

$H = 0$, so the dimensionless parameters Ω_m and Ω_Λ are ill-defined, but this point may be represented in an extended phase space as $\mathscr{E} = (\infty, \infty)$. This point is unstable, but is connected to the source **Bb** by an exceptional trajectory, which separates the models which recollapse from those which expand forever.

Restricting to $\Lambda = 0$, the only fixed points are **Bb** and **M**, with **Bb** a source and **M** a sink, see figure 3. The unstable Einstein-de Sitter universe **Bb** has slow volume growth $a \sim t^{2/3}$, while the stable Milne universe **M** has volume growth $a \sim t$. In fact, this growth rate is maximal among $\Lambda = 0$ models. This indicates that rapid volume growth goes together with stability.

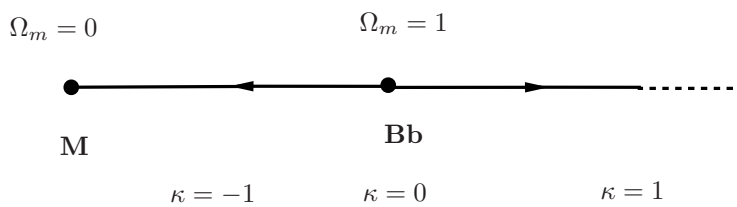


Fig. 3 The dynamics of Friedmann dust models for $\Lambda = 0$.

Now we can give an extremely simplified description of the current situation in cosmology by saying that the laws of general relativity together with the cosmological principle and observations leads to the "standard model" with the cosmological parameters

$$\Omega_{\kappa 0} \sim 0, \quad \Omega_{m0} \sim 0.3, \quad \Omega_{\Lambda 0} \sim 0.7, \quad h \sim 0.7.$$

The standard model is a big bang model. There is an initial singularity, $a \searrow 0$ as $t \searrow 0$ and the universe expands indefinitely to the future, $a \nearrow \infty$ as $t \nearrow \infty$. The model predicts a hot big bang, which leads to the prediction of cosmic background radiation [35, 36]. The observation of a highly homogenous cosmic background radiation with a spectrum close to that of a black body is a major success of the big bang models of cosmology.

Most of the energy density in the standard model consists at present of as yet unknown "dark matter" (accounting for approximately 85% of the matter density) and "dark energy" in the form of the cosmological constant. Dark matter, which for a long time has been broadly accepted in astronomy and cosmolocy, cf. [37], is distinguished from dark energy by the fact that its existence is motivated by studies of the dynamics of galaxy clusters and galactic rotation curves, which are independent of the Friedmann model which forms the basis of the standard model in cosmology. On the other hand, the cosmological constant was deemed unacceptable on philosophical grounds and entered the standard model fairly recently, shortly before the year 2000; the effects of dark energy being seen only indirectly via cosmological models and eg. studies of structure formation in the universe.

The acceptance of Λ came about only after the observation of the dimming of type Ia supernovae. The observations are interpreted as saying that the rate of expansion is accelerated, i.e. $\ddot{a} > 0$, which is incompatible with a Friedmann model

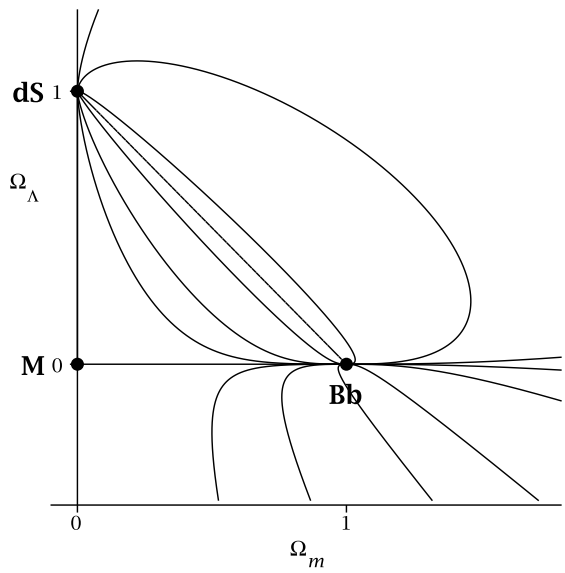


Fig. 4 This figure shows some orbits for Friedmann cosmologies with dust and dark energy (Λ) in the $(\Omega_m, \Omega_\Lambda)$ plane. The Einstein-de Sitter point $\mathbf{Bb} = (1,0)$ is a source, the Milne point $\mathbf{M} = (0,0)$ is a saddle node, and the de Sitter point $\mathbf{dS} = (0,1)$ is a sink. See [33] for background.

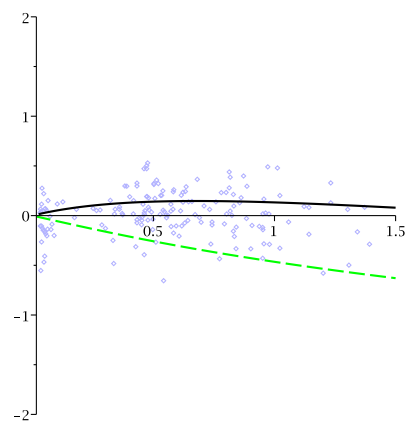


Fig. 5 Magnitude residual for SNe Ia Gold data [38] (dots) relative to the Milne model, plotted against redshift z. The black, solid curve is the standard model, while the green, dashed curve is the Einstein-de Sitter model. The horizontal axis is the Milne model.

filled with ordinary matter and $\Lambda = 0$. Figure 5 shows the supernova data compared to the standard model and Einstein-de Sitter. The horizontal axis is the Milne model ($\Omega_m = \Omega_\Lambda = 0$).

2.1 Cosmological problems

One of the important arguments against introducing the cosmological constant (apart from the difficulty of explaining the value Λ which appears motivated by cosmology from the point of view of particle physics) has been the *coincidence problem*, which might also be termed the "why now" problem. Figure 6 shows the time evolution of the dark energy density Ω_Λ. We see that it is only close to the present epoch that Ω_Λ becomes significant, and in the later universe it will dominate the dynamics. Due to the different scaling behavior of the matter and Λ densities in view of (3), the fact that these are both of order unity at the present epoch is a coincidence that could be argued to be contrary to the idea that we are not "special observers". In contrast, in the Einstein-de Sitter model the matter density is time independent.

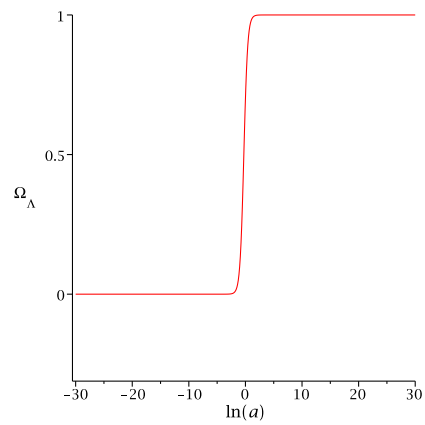


Fig. 6 The time evolution of the dark energy density, see [39] for discussion.

A related problem is the *flatness problem*. Roughly speaking, this is the question why $\Omega_\kappa \sim 0$ at present. In case $\Lambda = 0$ this can be seen to be problematic simply from figure 6. Since **Bb** is unstable, fine tuning of the initial conditions is required in order to have $\Omega_\kappa \sim 0$ at present. Lake [33] argues, using the presence of a conserved quantity for the dynamics in the $(\Omega_m, \Omega_\Lambda)$-plane, that fine tuning is not needed to have $\Omega_\kappa \sim 0$ throughout the history of the universe.

The universe is not exactly homogenous or isotropic; this holds at best in an approximate sense on sufficiently large scales. This raises the problem of whether it is possible to determine from observations, which are necessarily restricted to our past light cone, to what extent, and at what scales, the assumption of homogeneity and isotropy is valid. A problem here is that local isotropy (i.e. isotropy around the world line of one observer) does not imply global homogeneity.

The Ehlers-Gehren-Sachs theorem gives conditions under which it is possible to conclude from exact isotropy of the cosmic microwave background that the universe

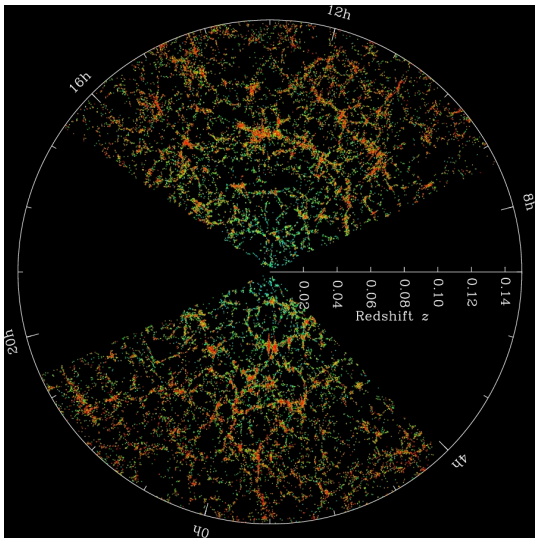


Fig. 7 Sloan Digital Sky Survey galaxy map, from www.sdss.org.

is exactly isotropic. However, this result can fail in several ways. For example, there are homogenous but non-isotropic models where the CMB is exactly isotropic at one instant in time. Extensions of the EGS theorem to situations where only approximate isotropy of the CMB holds are problematic, see [40, 41, 42] and references therein. This raises the problem of determining to what degree observations of the actual universe can be modelled and analyzed in the framework of Friedmann models (and perturbations thereof). One aspect of this problem is the question whether there is a scale at which (statistical) homogeneity and isotropy can be said to hold. Current estimates place this scale at approximately $150h^{-1}$ Mpc, see e.g. [43], see also [44]. However, recent observations indicate the existence of inhomogenous structures of a dimension which may be in conflict with isotropy at this scale, see [45]. It is conceivable that observations which extend to ever higher redshifts continue to yield evidence of structures in the universe of a size comparable to the homogeneity scale. Some aspects of inhomogeneity in cosmology were recently surveyed in a focus issue of CQG, see [46] and references therein.

The question of how the potential effects of large scale inhomogeneities on observations should be analyzed raises several important issues. Ellis has formulated the "fitting problem", see [47] and references therein, which asks about the effect of analyzing observations from an inhomogenous universe via a Friedmann model which is in some sense the "best fit" to the actual universe. The effect on observations of the fact that the model universe used to analyze data is only an approximation of the actual universe is sometimes referred to as "backreaction". An important question here is whether perturbation theory can be applied to take into account the deviation of the model from the actual universe. Kolb and collaborators have argued [48] that this analysis should take into account the peculiar velocities due to the different ex-

pansion rate in the model and the actual universe. Another effect of inhomogeneities which also sometimes is referred to as backreaction, is the *dynamical* effect of the inhomogeneities on the expansion of the universe. A possible approach is to use averaging [49] or coarse-graining [50] to derive a set of effective equations modelling the universe. In order to carry out such a scheme, one must introduce closure relations which allow one to extract an autonomous system. It is here worth mentioning the ideas on multi scale averaging, see e.g. [51, 52]. In particular, Wiltshire [53] argues that one should consider modifying the Copernican principle to take into account the idea that we reside in a gravitationally bound structure in a universe which has both bound systems and voids.

It is apparent that the matter distribution in the universe is "lumpy" due to the matter concentrations in stars, galaxies and other structures, and inhomogenous due to the presence of large scale voids and bound structures, and the effect of these must be taken into account when analyzing observations, see Clarkson et al. [54] for discussion. The optical properties of the universe are, in the Friedmann models which form the basis for the standard model of cosmology, calculated using the properties of a fluid which is used to approximate the actual matter distribution. Thus it is necessary to analyze whether the optical properties of a lumpy matter distribution differ in a significant way from the optical properties of a fluid. Light from distant stars passes through the gravitational wells of bound objects as well as voids on the way to the observer, and the effect of this process must be analyzed and compared to light passing through the fluid in a Friedmann model. This problem has been studied by among others Clifton et al [55], see also [56]. In this context, we also mention the so-called swiss cheese models, in which one attempts to analyze the optical effect of voids and structure in the universe by introducing under-densities in a background Friedmann model, see e.g. [57] and references therein. The swiss cheese models generally suffer from the limitation that the over-all expansion of the model is determined by the chosen background Friedmann geometry.

In this situation one may contemplate introducing weaker cosmological principles, incorporating ideas of statistical homogeneity, or weakening the Copernican principle by restricting to matter bound observers as suggested by Wiltshire.

As we have seen, the standard cosmological model is not located at a fixed point for the dynamical system governing the evolution of the dimensionless parameters Ω_m, Ω_Λ, rather it is close to the spatially flat orbit connecting the source **Bb** to the sink **dS**. Further, in that orbit, Ω_m/Ω_Λ takes on all positive real values. Thus, we as observers are not in an asymptotic regime, but rather, as mentioned above, at a special moment where Ω_m and Ω_Λ are both of order unity. Thus, from this point of view, we are neither in the "early universe" or the "late universe" and we cannot argue that our current universe is singled out as the asymptotic state of the evolution of the universe.

This makes the situation in cosmology rather different from the situation in many branches of physics where asymptotically stable objects are those which one expects to find in nature. As an example, the Kerr black hole solution is expected to be the unique stationary, asymptotically flat black hole spacetime. In order to establish the astrophysical significance of this solution, it is essential to prove that it is stable.

This leads to the black hole stability problem, one of the central open problems in general relativity. The problem of determining from observations whether or not for example the supermassive black holes expected to be found at the center of most galaxies are Kerr black holes or not is being actively studied.

As was just mentioned, from the point of view of the current standard model in cosmology, questions about the asymptotics of cosmological models do not appear to be the right ones to ask. Nevertheless, such questions give rise to interesting mathematical problems which we shall discuss in the rest of this paper. The questions about the asymptotic behavior of cosmological models include the structure of the big-bang singularity and questions about the behavior in the expanding direction. In particular we can ask: What does an observer in the late universe see?

3 Asymptotics of cosmological models

In this section we will describe a scenario for the asymptotic future behavior of cosmological models with vanishing cosmological constant. Recall that the Milne model with line element

$$ds^2 = -dt^2 + t^2 g_{\mathbb{H}^3}$$

where $g_{\mathbb{H}^3}$ is the hyperbolic 3-metric with sectional curvature -1, is isometric to the flat interior of the lightcone in Minkowski space. The Milne universe may be viewed as the future of O, the origin in Minkowski space. This point represents the big bang singularity in the Milne universe and is in the past of all spacetime points (i.e. all observers). The cosmological time at a point P is the proper time elapsed from the origin to P. The level surfaces of cosmological time are simply the hyperboloids. We next consider a flat, but non-isotropic model, which may be viewed as a deformation of Milne. Let I be a spacelike interval in Minkowski space and consider the future of I. The resulting spacetime can be constructed by cutting the Milne spacetime by a timelike hyperplane through O and gluing in a spacetime of the form $\mathbb{R}^{2+1} \times I$ with line element

$$-dt^2 + t^2 g_{\mathbb{H}^2} + dz^2 .$$

The deformed Milne spacetime has a big-bang singularity given by the interval I, and defining the cosmological time at P as the maximal proper time of any past inextendible geodesic starting at P the level sets of cosmological time are as in figure 8; it is flat and empty, but not homogenous and isotropic. Measuring the volume of co-moving regions in the deformed Milne universe we see that in the deformed regions, the volume of the cosmic time levels grows asymptotically as $t^{2/3}$, i.e. the growth rate of the Einstein-de Sitter universe, while in the undeformed regions, the growth rate is asymptotically as t. The behavior is similar for the level sets of the Hubble (mean curvature) time. On the other hand, asymptotically as $t \nearrow \infty$, the volume fraction in the undeformed region tends to 1, while in the asymptotic past (near the big bang) these regions have a negligible volume fraction.

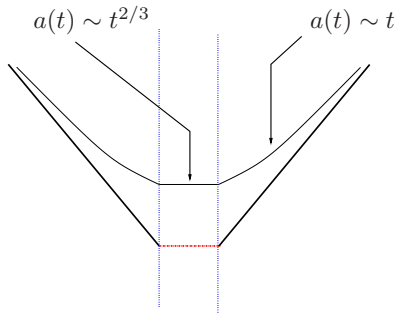


Fig. 8 A flat cosmological spacetime not isometric to Milne. A level set of cosmological time t is shown. The vertical lines indicate the flat wedge which has been glued in.

More general flat spacetimes may be constructed as the future of sets (e.g. fractals) in Minkowski space, and quotients of these by the action of discrete groups of isometries. Flat, or more generally, constant curvature spacetimes are examples of G-structures and such spacetimes admitting compact Cauchy surfaces have been completely analyzed, starting with the work of Mess [58], see also [59], who analyzed the class of constant curvature 2+1 dimensional spacetimes admitting a compact Cauchy surface. For example, one may show that the space of flat 2+1 dimensional spacetimes with Cauchy surface of genus $g > 1$ is isomorphic to $\partial\mathscr{M} \times \mathscr{M}$, where $\mathscr{M}$ is Teichmuller space of surfaces of genus g and $\partial\mathscr{M}$ is the Thurston boundary. The particular case of constant curvature spacetimes with compact Cauchy surface has been analyzed in [60]. In particular, it was shown there that such flat spacetimes can be globally foliated by Cauchy surfaces of constant mean curvature (i.e. constant Hubble time).

The level sets of Hubble time can be related to the level sets of the cosmological time by an application of a maximum principle, and one may show that the volume growth of these level sets is comparable to that of the level sets of the cosmological time. This leads to a generalization of the statements made above for the simple deformed Milne universe, see [61].

In view of the above mentioned work, these generalized Milne spacetimes may have a very complex (e.g. fractal) big bang type initial singularity. In some cases their future asymptotics can be analyzed, see [62]. One finds that the level sets of Hubble time decompose into "neck regions" with slow volume growth, and "hyperbolic regions" with fast volume growth. The scale free geometry of these level sets may may be depicted as in figure 9.

In particular, one finds that in the asymptotically expanding direction, the volume fraction of asymptotically, hyperbolic (thick) regions dominate while the neck regions (thin) become insignificant. Therefore, a "typical" (volume averaged) observer at late time lives in a thick region.

It is interesting to compare the relation between the thin and thick regions to the overdense and void regions in an inhomogenous universe containing matter, in

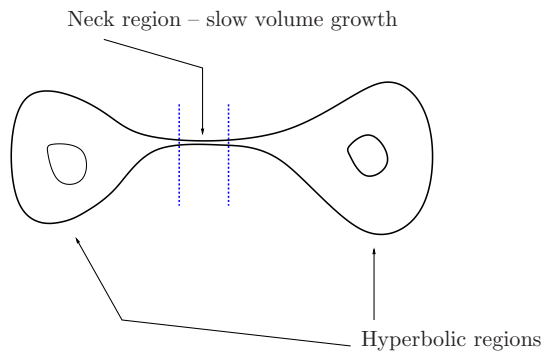


Fig. 9 Qualitative shape of Hubble level set.

particular in view of the fact that the thin regions have volume growth approximating that of Einstein-de Sitter universe which has critical matter density.

We now consider the generalization of the above picture to the case of general, inhomogenous universes. We start by noting that the Lorentzian Einstein equations define a flow on the space of (scale free) geometries. By analogy with the Ricci flow of Riemannian geometries, this may be termed the Einstein flow.

For simplicity, we consider spacetimes $(\mathbf{M}, \mathbf{g}_{ab})$ of dimension $D = d+1$ which are vacuum, i.e. with

$$\mathbf{R}_{\alpha\beta} = 0\,.$$

Suppose $\mathbf{M}$ admits a foliation by Cauchy surfaces of constant mean curvature H. Introduce the dimensionless logarithmic constant mean curvature (Hubble) time $T = -\ln(H/H_0)$, and consider the evolution of the scale free geometry $[g] = H^2 g$. The Lorentzian Einstein equations define a flow $T \mapsto [g](T)$, on the space of scale free geometries. In particular, in the 2+1 dimensional case, the Einstein equations correspond to a time dependent Hamiltonian system on Teichmüller space [63], and each universe corresponds to a curve connecting a point on the boundary of Teichmuller space to an interior point, see figure 10.

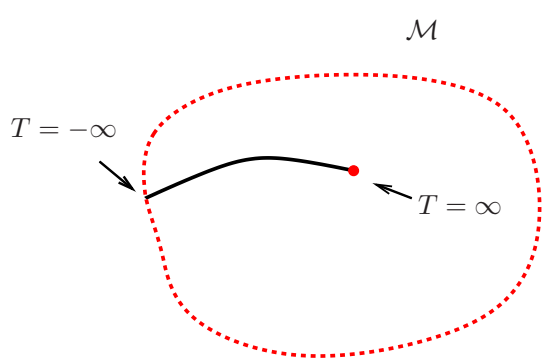


Fig. 10 The Einstein flow in the 2+1 dimensional case.

One arrives at the following heuristic scenario [64, 65]. Consider spacetimes with Cauchy surface M. The non-collapsing case corresponds to the case where M has negative Yamabe type. For $T \nearrow \infty$, $(M,[g])$ decomposes into hyperbolic pieces and Seyfert fibered pieces, and this decompsition corresponds to a (weak) geometrization, cf. [66]. The Einstein flow in CMC time results in a thick/thin decomposition of M, where the thick (hyperbolic) pieces have full volume growth. As a consequence we have that in the far future, the hyperbolic pieces represent most of the volume of M, cf. figure 11. Proving statements along the lines described above appears to be very difficult, and one must therefore start by considering sub-problems.

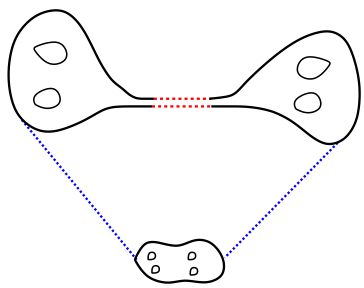


Fig. 11 The collapse of necks in the Einstein flow.

4 Results on nonlinear stability

To give some perspective on the nonlinear stability problems introduced above, we discuss some results on other stability problems in general relativity. These are organized according to the asymptotic model spacetime. The black hole stability problem, cf. [67] for discussion and references, is not mentioned here. In the following, we mention only the cases with conformally flat background spacetimes.

4.1 Minkowski

First we consider the nonlinear stability of Minkowski space, i.e. $\mathbb{R}^4$ with line element

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2.$$

The conformal type of Minkowski space is that of the Minkowski diamond, see figure 12. In this causal diagram, each interior point represents a 2-sphere.

Nonlinear stability holds, in the sense that for Cauchy data near Minkowski data, the maximal development is geodesically complete and asymptotically Minkowskian.

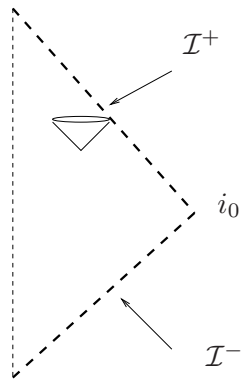


Fig. 12 Conformal diagram of Minkowski space.

A key fact is that radiation carries energy through the conformal boundary $\mathscr{I}$. Due to the fact that the nonlinearity in the Einstein equations is quadratic, it is necessary to exploit a cancellation in the equations in order to prove stability.

The first result in this direction is due to H. Friedrich [68], who proved that for data close to the data induced on a hyperboloid in Minkowski space, one has nonlinear stability to the future, and with suitable asymptotic regularity for the data, the maximal development has a regular $\mathscr{I}^+$ to the future of the initial slice. The full nonlinear stability result was proved by Christodoulou and Klainerman [69]. This work was extended to include the full peeling at $\mathscr{I}$ by Klainerman and Nicolo [70]. A simpler proof of nonlinear stability, using wave coordinates (spacetime harmonic coordinates) gauge was given by Lindblad and Rodnianski [71]. Using both of these methods, the proof of nonlinear stability can be readily adapted to the Einstein-matter system, provided that the matter fields do not destroy the conformal properties of the Einstein equations. Examples include a massless scalar field, which was included in the work of Lindblad and Rodnianski, and a Maxwell field, see [72].

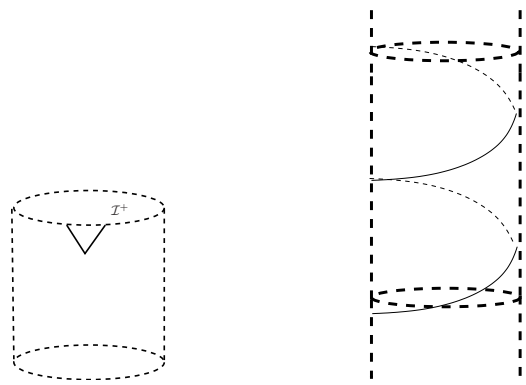
4.2 de Sitter

Next we consider cosmological models with positive Λ. The canonical example is de Sitter space with line element

$$ds^2 = -dt^2 + \cosh^2(t) g_{S^3} .$$

This is conformal to a finite cylinder with spacelike conformal boundary, and hence one has future horizons and "locality" at $\mathscr{I}^+$. Due to this fact, topology does not matter for the future dynamics (but cf. [73]). Due to the locality at $\mathscr{I}^+$, we have that a suitable notion for smallness in the stability argument can be defined locally in

space. We mention some results in this setting. H. Friedrich proved global nonlinear stability of de Sitter space for the Einstein-Yang-Mills system with positive cosmological constant [74]. H. Ringström proved a "local in space" small data global existence results for the Einstein-Λ-scalar field system [75, 76]. The case of fluid matter was considered in this situation by Rodnianski and Speck [77] for the irrotational case, see Speck [78] for the Einstein-Euler system. Finally, the Einstein-Λ-Vlasov system has been studied by Ringström [79].



deSitter spacetime: conformal to a finite cylinder.

Milne spacetime: conformal to an infinite cylinder.

Fig. 13 Conformal diagrams of deSitter and Milne spacetimes.

4.3 Milne

Finally we consider the stability problem for a cosmological models with $\Lambda = 0$. Here, the only general results are for the vacuum case. By passing to a quotient of the Milne spacetime, we may consider a flat spacetime which has a Cauchy surface isometric to a compact hyperbolic 3-manifold. The line element is

$$ds^2 = -dt^2 + t^2 g_{\mathbb{H}^3}$$

($\kappa = -1$ empty Friedmann) and the spacetime is conformal to an infinite cylinder

$$-d\tau^2 + g_{\mathbb{H}^3}$$

In this case topology does matter, in the sense that an observer is able to see the whole past of his spacetime. Since there is no future conformal boundary, it is not possible to localize the future evolution problem.

Future stability for Milne with compact Cauchy surface as described above was proven by the author in collaboration with Moncrief for spacetime dimension $d+1$, $d \geq 3$, cf. [80, 81], see also [82, 83]. For the 2+1 dimensional case, see [63]. Concerning the stability problem for the Einstein-matter systems in this setting, much less is known than in the case with positive Λ. Some sub-problems have been considered for the Einstein-Vlasov system in Bianchi symmetry (spacetimes with a 3-dimensional Lie group acting by isometries on Cauchy surfaces), see [84], [85], [86]. Finally, we mention the work concerning test fluids on Friedmann backgrounds by J. Speck [87].

The case of vacuum spacetimes with $U(1)$ symmetry leads after a Kaluza-Klein reduction to 2+1 dimesional gravity with wave maps matter. The non-linear stability of the flat cones over surfaces of genus $g > 1$ in this setting has been studied by studied by Choquet-Bruhat and Moncrief, see [88, 89].

5 Generalized Kasner spacetimes

In section 4.3 we discussed a stability theorem for the future of a Cauchy surface in a class of spacetimes. The background spacetime in that case is a Lorentz cone over a compact Einstein space with negative scalar curvature, i.e. a generalized Milne space. In particular these are warped products of the line with an Einstein space. In this section we shall discuss a class of double warped product spacetimes, with two scale factors. These spacetimes which were considered in [90] may be viewed as generalized Kasner spacetimes. They have the form

$$\mathbf{M} \cong \mathbb{R} \times M \times N\,,$$

with (M,g), (N,h), compact negative Einstein spaces of dimensions m, n, respectively. The dimension of $\mathbf{M}$ is $D = d+1 = m+n+1$. We assume $\mathrm{Ric}_g = -(m+n-1)g$, $\mathrm{Ric}_h = -(m+n-1)h$. and consider a line element on $\mathbf{M}$ of the form

$$ds^2 = -dt^2 + a^2(t)g + b^2(t)h\,.$$

Let $p = -\dot{a}/a, \quad q = -\dot{b}/b$, and introduce the scale invariant variables

$$P = p/H, \quad Q = q/H, \quad A = \frac{1}{aH}, \quad B = \frac{1}{bH}\,.$$

The Einstein equations imply an autonomous system for (P,Q,A,B) with 2 constraints. A dynamical systems analysis shows that the generic orbit has generalized Kasner behavior, i.e. $a \sim t^p$, $b \sim t^q$ at singularity, and is asymptotically Friedmann (in fact asymptotic to a Lorentz cone spacetime) in the expanding direction

$$a,b = t + O(t^{1-\lambda^*}), \quad \lambda^* > 0\,.$$

Friedmann is a stable node only in spacetime dimension $D \geq 11$.

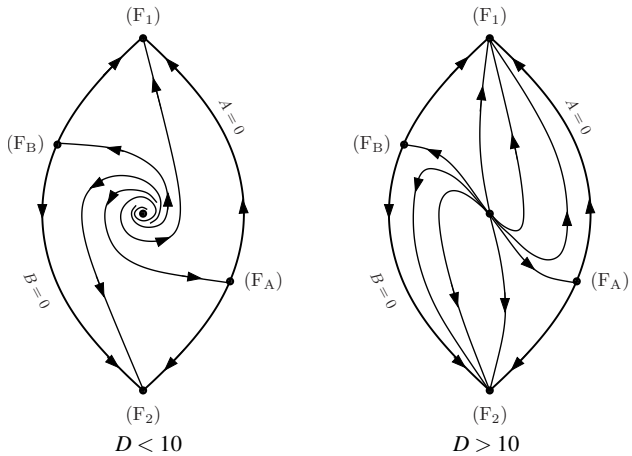


Fig. 14 The dynamics of the generalized Kasner models [90]. The arrows point in the past direction. There are five fixed points, one of which is the Friedmann point in the interior of the phase space. The Friedmann point is a past unstable node for $D > 10$ and a unstable spiral point for $D < 10$. The past stable fixed points $F_{1,2}$ satisfy condition (6) for $D > 10$. This implies quiescent behavior at the singularity for inhomogenous deformations of the generalized Kasner models in $D > 10$.

5.1 From α to ω

Belinskiĭ, Khalatnikov, and Lifshitz, [91] argued that a generic cosmological singularities in 3+1 dimensions in spacetimes with ordinary matter is oscillatory. The picture developed by Belinskiĭ, Khalatnikov, and Lifshitz is often referred to as the BKL proposal. BKL type behavior has been proved rigorously so far only for the Bianchi VIII and IX models, see [92], where also strong cosmic censorship for this class of models was shown. On the other hand, Belinskiĭ and Khalatnikov [93] pointed out that cosmological singularities in spacetimes containing stiff fluid or scalar field can be non-oscillatory, or *quiescent*. The heuristic analysis of Belinskiĭ and Khalatnikov was extended to the higher dimensional case by Demaret et al. [94] who showed that quiescent behavior at singularity in $D = d + 1$ dimensions holds if the condition

$$1 + p_1 - p_d - p_{d-1} > 0 \tag{6}$$

holds, where p_a are the generalized Kasner exponents at the singularity. This heuristic analysis shows that (6) holds in vacuum *only* if $D \geq 11$, and hence one expects that generic vacuum, $D < 11$ spacetimes have oscillatory singularity, while generic vacuum, $D \geq 11$ spacetime have quiescent singularity. It was shown in [90, §4] that (6) holds for generalized Kasner spacetimes if $D \geq 11$, in agreement with the result of [94].

As a step towards making this heuristic scenario rigorous, the author showed with Rendall [95] that generic $D = 4$ spacetime with scalar field has quiescent singularity. In that paper we constructed a full parameter family of Einstein-scalar field and

Einstein-stiff fluid spacetimes with quiescent singularity using Fuchsian analysis. This work was extended to the case of $D \geq 11$ vacuum spacetimes by Damour et al. [96], again using a Fuchsian analysis.

One may use the techniques discussed above to prove that a type of global nonlinear stability holds for a class of generalized Kasner spacetimes. It was shown in [97] that for generalized Kasner spacetimes as above, with $D \geq 11$, satisfying the additional condition that the moduli space of negative Einstein metrics on M, N is integrable (which is expected to hold in general), there is a full-parameter family of C^{ω} Cauchy data on $M \times N$, such that the maximal Cauchy development $(\mathbf{M}, \mathbf{g})$ has a global CMC time function, and has quiescent, crushing singularity. Further $(\mathbf{M}, \mathbf{g})$ is future causally complete and is asymptotically Friedmann to the future, with $g(T) \to \gamma_{\infty}^{M} + \gamma_{\infty}^{N}$, as $T \to \infty$, where γ_{∞}^{M} and γ_{∞}^{N} are negative Einstein metrics on M, N, respectively. This applies to a large variety of factors M, N, and can easily be generalized to multiple factors.

6 Concluding remarks

In this paper we have given brief overview of some of the ideas underlying the general relativistic cosmological models which form the core of the standard model of cosmology, and pointed out the need for an improved analysis, both from the physical and mathematical point of view, of the effect of deviations from homogeneity and isotropy in the dynamics of cosmological models, and consequently in the analysis of cosmological data. Motivated by this, we have discussed some results on nonlinear stability for cosmological models. We end by listing some open problems.

The exponential expansion caused by the presence of the cosmological constant in the case $\Lambda > 0$ and also in the presence of certain self-gravitating scalar field models for inflation makes the large data future behavior of these models tractable and here there are several results which do not require any symmetry assumtions, see section 4.2.

For the case $\Lambda = 0$ and ordinary matter, the situation is more delicate. The global behavior of cosmological models is well understood in highly symmetric cases, including the 3+1 dimensional Friedmann, Bianchi, Gowdy (spatial T^2 symmetric, with symmetry action generated by hypersurface orthogonal Killing fields) and so-called surface symmetric cases, see [98] and references therein. For the Bianchi case, see the remarks in section 5.1 and [99, 100], and for the Gowdy case see [101] and references therein. However, for large data, the asymptotic behavior of the general T^2, $U(1)$ (circle symmetric) and the full 3+1 case are mostly open. Similarly, future stability is open in the 3+1 dimensional case for Einstein-matter models without symmetry assumptions in the case $\Lambda = 0$. As an example, one would like to prove nonlinear stability of Milne for Einstein-Vlasov. This is work in progress by the author with D. Fajman.

Our understanding of the behavior of cosmological models in the direction of the initial singularity is also limited. The BKL proposal provides a heuristic scenario which has been verified only in the Bianchi case, where also strong cosmic censorship has been shown to hold, see above. However, in spite of some recent progress [102, 103, 104], even the question whether the singularity in generic Bianchi models is local, is open. See [105, 100] for references and discussion. For Gowdy models with T^3 Cauchy surface, Ringström has proved that strong cosmic censorship holds, see [101] for an overview, while for Gowdy with Cauchy surfaces diffeomorphic to S^3 or $S^2 \times S^1$, and the general T^2 symmetric case (dropping the condition on hypersurface orthogonality) the situation is much more complicated and cosmic censorship is open. In particular, in the T^2 symmetric case, one has the new phenomenon of dynamical spikes, see [106, 107].

The work by the author and Rendall, and by Damour et al. on quiescent singularities, see section 5.1 opens up the problem of proving quiescent behavior at the singularity as well as global nonlinear stability for an open set of Cauchy data (in a suitable topology). This is work in progress by the author and Ringström. Work on this type of stability problem for the Friedmann case was mentioned in a recent talk by Speck [108]. For the case $D < 11$ one may consider suitable Einstein-scalar field models and for $D \geq 11$ one may formulate the global nonlinear stability problem for the generalized Kasner backgrounds as discussed in section 5. Here it should be pointed out that the global stability result mentioned there relies on Fuchsian methods and therefore suffers from the same weakness as the work by the author and Rendall, and Damour et al. on quiescent singularities. It would be interesting to prove a true nonlinear stability result, stating that for an open set of Cauchy data close to the generalized Kasner background data, the maximal development is geodesically complete to the future, asymptotically Friedmann, and with crushing singularity with geometry close to, in a suitable sense, the singularity in the generalized Kasner spacetime.

For the near future, I expect that numerical studies of cosmological models in GR, with less symmetry than the 2 Killing field models including LTB, T^2 and spherically symmetric models studied in detail so far, will play an important role in exploring the future behavior of cosmological models. One can expect that such investigations will have an impact on both physical cosmology and the mathematical analysis of cosmological models.

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