Observers, observables and measurements in general relativity

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Abstract To perform any physical measurement it is necessary to identify in a non ambiguous way both the observer and the observable. A given observable can be then the target of different observers: a suitable algorithm to compare among their measurements should necessarily be developed, either formally or operationally. This is the task of what we call "theory of measurement," which we discuss here in the framework of general relativity.

1 Introduction

The spacetime (or absolute) point of view constitutes a unified scenario for quantities which, in the pre-relativistic physics, were associated with distinct notions: time and space themselves, energy and momentum, mechanical power and force, electric and magnetic fields and so on. In every day experience, however, our intuition is still compatible with the perception of a three-dimensional space and a one-dimensional time and therefore any physical measurement requires a local recovery of the pre-relativistic type of separation between space and time. To this purpose we need some prescription in order to perform the required splitting, and hence identifying a "space" and a "time" relative to any given observer. Any such prescription requires a congruence of timelike world lines with a future-pointing unit tangent vector field u (i.e., the local time direction) which we interpret as the world lines of a family of (test) observers with associated 4-velocity u.

The splitting of the tangent space at each point of the congruence into the local time direction u and the local rest space spanned by vectors orthogonal to u (hereafter LRS_u), allows one to decompose all spacetime tensors, including tensorial operators, and tensor equations into their spatial and temporal components. One

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may ask then if there exist natural or special observer families in a given spacetime. This is clearly the case of a stationary spacetime where a special observer (timelike) congruence is associated with the timelike Killing vector field. Also, any spacetime admitting a spacelike foliation has naturally associated with it a timelike congruence, namely that of the normal directions to the slicing itself. It is known that any spacetime admitting separable geodesics (e.g., as a consequence of the existence of a Killing tensor of rank 2) also admits a foliation. For example, Kerr spacetime with the metric written in standard Boyer-Lindquist coordinates summarizes these three conditions: it has the family of static observers (whose world lines are aligned with the coordinate time lines, i.e., with unit tangent vector parallel to the timelike Killing vector ∂_t), the family of locally nonrotating observers or ZAMOs (whose world lines are orthogonal to the t=constant hyersurfaces) and finally, the families of Painlevé-Gullstrand observers who follow geodesic timelike lines since the latter have a separable dependence from the coordinates. A review of the essential splitting formalism follows below. For more details one can refer for instance to Refs. [1, 2, 3, 4, 5, 6] (and references therein).

2 Orthogonal decompositions

Let g be the four-dimensional spacetime metric with signature $+2$ and components $g_{\alpha\beta}$ $(\alpha,\beta=0,1,2,3)$, ∇ its associated covariant derivative operator and η the unit volume 4-form which assures spacetime orientation. Let u be a future-pointing unit timelike vector field which identifies an observer, $u\cdot u=-1$. The local splitting of the tangent space into orthogonal sub-spaces uniquely related to the given observer u, is accomplished by a temporal projection operator $T(u)$ (along u) and a spatial projection operator $P(u)$ (generating the LRS_u). These operators, in mixed form, are defined as follows

$$T(u)=-u^\sharp\otimes u^\flat \quad P(u)=I+u^\sharp\otimes u^\flat \tag{1}$$

where $I\equiv\delta^\alpha{}_\beta$ is the identity on the tangent spaces of the manifold and the symbols $\sharp$ and $\flat$ identify the fully contravariant and covariant representation of tensors, respectively. In terms of components the above relations write

$$T(u)^\alpha{}_\beta=-u^\alpha u_\beta\ ,\quad P(u)^\alpha{}_\beta=\delta^\alpha{}_\beta+u^\alpha u_\beta\,. \tag{2}$$

Given a $\binom{p}{q}$-tensor S, let us denote as $[P(u)S]$ its fully spatial projection obtained by acting with the operator $P(u)$ on all of its indices,

$$[P(u)S]^{\alpha\cdots}{}_{\beta\ldots}=P(u)^\alpha{}_\gamma\cdots P(u)^\delta{}_\beta\cdots S^{\gamma\ldots}{}_{\delta\ldots} \tag{3}$$

The splitting of S relative to a given observer is the set of tensors which arise from the spatial and temporal projection of each of its indices as we are going to discuss.

This observer-dependent set of tensors represent S and it is termed as its (geometrical) measurement by the observer u.

1. Splitting of a vector
   If S is a vector field then its splitting gives rise to a scalar field and a spatial vector field

$$S \quad \leftrightarrow \quad \{u \cdot S, [P(u)S]\} \ . \tag{4}$$

   In terms of components they read

$$S^\alpha \quad \leftrightarrow \quad \{u_\gamma S^\gamma, P(u)^\alpha{}_\gamma S^\gamma\} \, . \tag{5}$$

   In fact, with respect to the observer u, the vector S admits then the following representation

$$S^\alpha = [T(u)S]^\alpha + [P(u)S]^\alpha = -(u_\gamma S^\gamma) u^\alpha + P(u)^\alpha{}_\gamma S^\gamma \ . \tag{6}$$

2. Splitting of a $\binom{1}{1}$-tensor
   If S is a mixed $\binom{1}{1}$-tensor field, then its splitting consists of a scalar field, a spatial vector field, a spatial 1-form and a spatial $\binom{1}{1}$-tensor field, namely

$$S^\alpha{}_\beta \leftrightarrow \{u^\delta u_\gamma S^\gamma{}_\delta, \, P(u)^\alpha{}_\gamma u^\delta S^\gamma{}_\delta, \, P(u)^\delta{}_\alpha u_\gamma S^\gamma{}_\delta, \, P(u)^\alpha{}_\gamma P(u)^\delta{}_\beta S^\gamma{}_\delta\} \ .$$

   In terms of these fields, the tensor S admits the following representation with respect to the observer u

$$\begin{aligned} S^\alpha{}_\beta &= [T(u)^\alpha{}_\gamma + P(u)^\alpha{}_\gamma][T(u)^\delta{}_\beta + P(u)^\delta{}_\beta] S^\gamma{}_\delta \\ &= (u^\delta u_\gamma S^\gamma{}_\delta) u^\alpha u_\beta - u^\alpha u_\gamma P(u)^\delta{}_\beta S^\gamma{}_\delta \\ &\quad - u^\delta u_\beta P(u)^\alpha{}_\gamma S^\gamma{}_\delta + [P(u)S]^\alpha{}_\beta \ . \end{aligned} \tag{7}$$

   The local spatial and temporal projections of a $\binom{p}{q}$-tensor is easily generalized. For example, the metric tensor $g_{\alpha\beta}$ has the (trivial) representation

$$g_{\alpha\beta} = P(u)_{\alpha\beta} + T(u)_{\alpha\beta}.$$

3. Splitting of p-forms
   Given a p-form

$$S = S_{[\alpha_1 \ldots \alpha_p]} \omega^{\alpha_1} \otimes \ldots \otimes \omega^{\alpha_p} \equiv \frac{1}{p!} S_{\alpha_1 \ldots \alpha_p} \omega^{\alpha_1} \wedge \ldots \wedge \omega^{\alpha_p} \tag{8}$$

   we define the electric part of S relative to the observer u the quantity

$$\left[S^{(\mathrm{E})}(u) \right]_{\alpha_1 \ldots \alpha_{p-1}} = -u^\sigma S_{\sigma \alpha_1 \ldots \alpha_{p-1}} \tag{9}$$

or in a more compact form $S^{(\mathrm{E})}(u) = -u \rfloor S$. Similarly we define as the magnetic part of S the quantity

$$\left[S^{(\mathrm{M})}(u)\right]_{\alpha_1\ldots\alpha_p} = P(u)^{\beta_1}{}_{\alpha_1}\ldots P(u)^{\beta_p}{}_{\alpha_p} S_{\beta_1\ldots\beta_p} \tag{10}$$

or, in a compact form, $S^{(\mathrm{M})}(u) = P(u)S$. From the above definitions we deduce the following representation of S

$$S = u^\flat \wedge S^{(\mathrm{E})}(u) + S^{(\mathrm{M})}(u)\ , \tag{11}$$

or in components

$$S_{\alpha_1\ldots\alpha_p} = p!u_{[\alpha_1}[S^{(\mathrm{E})}(u)]_{\alpha_2\ldots\alpha_p]} + [S^{(\mathrm{M})}(u)]_{\alpha_1\ldots\alpha_p}\ . \tag{12}$$

For example, the splitting of the unit volume 4-form η gives rise to the following representation

$$\eta = -u^\flat \wedge \eta(u)\,, \tag{13}$$

that is $[u^\flat \wedge \eta(u)]_{\alpha\beta\gamma\delta} = \left[2u_{[\alpha}\eta(u)_{\beta]\gamma\delta} + 2u_{[\gamma}\eta(u)_{\delta]\alpha\beta}\right]$, where the spatial unit volume 3-form

$$\eta(u)_{\alpha\beta\gamma} = u^\delta \eta_{\delta\alpha\beta\gamma} \tag{14}$$

is the only nontrivial spatial field which arises from the splitting of the volume 4-form. Using the spacetime (Hodge) duality operation (*), one can associate with any p-form S (with $0 \le p \le 4$) a $(4-p)$-form. Similarly a spatial duality operation ($^{*(u)}$) is defined for a spatial p-form S ($u \rfloor S = 0$) replacing η with $\eta(u)$, namely

$$^{*(u)}S_{\alpha_1\ldots\alpha_{3-p}} = \frac{1}{p!} S_{\beta_1\ldots\beta_p}\eta(u)^{\beta_1\ldots\beta_p}{}_{\alpha_1\ldots\alpha_{3-p}}\ . \tag{15}$$

For example, given a spatial 2-form S, its spatial dual is

$$[^{*(u)}S]^\alpha = \frac{1}{2}\eta(u)^{\alpha\beta\gamma}S_{\beta\gamma}. \tag{16}$$

This operation satisfies the property $^{*(u)}{}^{*(u)}S = S$. Let us now consider the splitting of $^*S$ where S is given by (11). We have

$$\begin{aligned} ^*S &= u^\flat \wedge [^*S]^{(\mathrm{E})}(u) + [^*S]^{(\mathrm{M})}(u) \\ &= {}^*[u^\flat \wedge S^{(\mathrm{E})}(u) + S^{(\mathrm{M})}(u)] \\ &= {}^{*(u)}S^{(\mathrm{E})}(u) + {}^*[^{*(u)}[^{*(u)}[S^{(\mathrm{M})}(u)]]] \\ &= {}^{*(u)}S^{(\mathrm{E})}(u) + (-1)^{p-1}u^\flat \wedge {}^{*(u)}[S^{(\mathrm{M})}(u)]\ . \end{aligned} \tag{17}$$

Comparing the first and the last line we have

$$[^*S]^{(\mathrm{E})}(u) = (-1)^{p-1}{}^{*(u)}[S^{(\mathrm{M})}(u)], \quad [^*S]^{(\mathrm{M})}(u) = {}^{*(u)}S^{(\mathrm{E})}(u)\ . \tag{18}$$

4. Splitting of differential operators
   In general relativity one has several spacetime tensorial differential operators which act on tensor fields. Let us recall them: if T is a tensor field of any rank, we have

   a. The Lie derivative of T along the direction of a given vector field X: $[\pounds_X T]$.

   b. The covariant derivative of T: ∇T.

   c. The absolute derivative of T along a curve with unit tangent vector X and parameterized by s: $\nabla_X T \equiv DT/ds$.

   d. The Fermi-Walker derivative of T along a non-null curve with unit tangent vector X and parameterized by s defined by

   $$\frac{D_{(\mathrm{fw},X)}T^\alpha{}_\beta}{ds} = \frac{DT^\alpha{}_\beta}{ds} \pm \left([a(X)\wedge X]^\alpha{}_\gamma T^\gamma{}_\beta - [a(X)\wedge X]^\gamma{}_\beta T^\alpha_\gamma\right) ,$$

   where $\pm$ refer to transport along timelike or spacelike curves, respectively.

   Finally if S is a p-form, one has

   (v) The exterior derivative of S: dS.

   Application of the spatial projection into the LRS_u of a family of observers u to the spacetime derivatives (i) to (v), yields new operators which can be more easily confronted with those defined in a three-dimensional Euclidean space. Given a tensor field T of components $T^{\alpha\ldots}{}_{\beta\ldots}$ we have in fact

   a. The spatially projected Lie derivative along a vector field X

   $$[\pounds(u)_X T]^{\alpha\ldots}{}_{\beta\ldots} \equiv P(u)^\alpha{}_\sigma \ldots P(u)^\rho{}_\beta \ldots [\pounds_X T]^{\sigma\ldots}{}_{\rho\ldots}; \tag{19}$$

   when $X = u$ we use also the notation

   $$\nabla(u)_{(\mathrm{lie})}T \equiv \pounds(u)_u T \ , \tag{20}$$

   and this operation will be termed “spatial-Lie temporal derivative”.
   b. The spatially projected covariant derivative along any e_γ frame direction

   $$\nabla(u)_\gamma T \equiv P(u)\nabla_\gamma T, \tag{21}$$

   namely

   $$[\nabla(u)_\gamma T]^{\alpha\ldots}{}_{\beta\ldots} = P(u)^\alpha{}_{\alpha_1} \ldots P(u)^{\beta_1}{}_\beta \ldots P(u)^\sigma{}_\gamma \nabla_\sigma T^{\alpha_1\cdots}{}_{\beta_1\ldots}.$$

   c. The spatially projected absolute derivative along a curve with unit tangent vector X

$$[P(u)\nabla_X T]^{\alpha\dots}{}_{\beta\dots} = P(u)^\alpha{}_{\alpha_1}\dots P(u)^{\beta_1}{}_\beta \dots [\nabla_X T]^{\alpha_1\dots}{}_{\beta_1\dots} \tag{22}$$

d. The spatially projected "Fermi-Walker derivative" along a curve with unit tangent vector X and parameterized by s

$$\left[P(u)\frac{D_{(\mathrm{fw},X)}T}{ds}\right]^{\alpha\dots}{}_{\beta\dots} = P(u)^\alpha{}_\sigma \dots P(u)^\rho{}_\beta \dots \left[\frac{D_{(\mathrm{fw},X)}T}{ds}\right]^{\sigma\dots}{}_{\rho\dots} .$$

e. the spatially projected exterior derivative of a p-form S

$$d(u)S \equiv P(u)dS, \tag{23}$$

namely $[d(u)S]_{\alpha_1\dots\alpha_p\beta} = P(u)^{\beta_1}{}_{\alpha_1}\dots P(u)^\sigma{}_\beta [dS]_{\beta_1\dots\sigma}$.

Note that all these spatial differential operators are well defined since they arise from the spatial projection of spacetime differential operators. From their definitions it is clear that both the Fermi-Walker and the Lie derivatives of the vector field u along itself vanish identically (and so do the projections orthogonal to u of these derivatives). The only derivative of u along itself which is meaningful being different than zero is the covariant derivative

$$P(u)\nabla_u u = \nabla_u u = a(u)\ . \tag{24}$$

3 Three-dimensional notation

Let u be a given a family of observers and X a spatial vector with respect to u. It is then convenient to introduce the 3-dimensional vector notation for the spatial inner product and the spatial cross product of two spatial vector fields X and Y. The spatial inner product is defined as

$$X \cdot_u Y = P(u)_{\alpha\beta}X^\alpha Y^\beta \tag{25}$$

while the spatial cross product is

$$[X \times_u Y]^\alpha = \eta(u)^\alpha{}_{\beta\gamma}X^\beta Y^\gamma\ , \tag{26}$$

where $\eta(u)^\alpha{}_{\beta\gamma} = u_\sigma \eta^{\sigma\alpha}{}_{\beta\gamma}$ as stated.

In terms of the above definitions we can define spatial gradient, curl and divergence operators of functions f and spatial vector fields X as

$$\mathrm{grad}_u f = \nabla(u)f\ , \quad \mathrm{curl}_u X = \nabla(u)\times_u X\ , \quad \mathrm{div}_u X = \nabla(u)\cdot_u X\ . \tag{27}$$

In components these relations read

$$
\begin{aligned}
[\mathrm{grad}_u f]_\alpha &= \nabla(u)_\alpha f = P(u)^{\alpha\beta} e_\beta(f) \ , \\
[\mathrm{curl}_u X]^\alpha &= \eta(u)^{\alpha\beta\gamma} \nabla(u)_\beta X_\gamma = u^\sigma \eta_\sigma{}^{\alpha\beta\gamma} \nabla_\beta X_\gamma \ , \\
[\mathrm{div}_u X] &= \nabla(u)_\alpha X^\alpha = P(u)^{\alpha\beta} \nabla_\alpha X_\beta \ .
\end{aligned} \tag{28}
$$

It is useful to extend the above definitions to

1. the spatial cross product of a vector X by a symmetric tensor A,

$$
[X \times_u A]^{\alpha\beta} = \eta(u)^{\gamma\delta(\alpha} X_\gamma A^{\beta)}{}_\delta \ , \tag{29}
$$

2. the spatial cross product of two symmetric spatial tensors A and B,

$$
[A \times_u B]_\alpha = \eta(u)_{\alpha\beta\gamma} A^\beta{}_\delta B^{\delta\gamma} \ , \tag{30}
$$

3. the spatial inner product of two symmetric spatial tensors A and B,

$$
[A \cdot_u B]_\alpha{}^\beta = A_{\alpha\gamma} B^{\gamma\beta} \ , \tag{31}
$$

4. the trace of the above tensor product

$$
\mathrm{Tr}[A \cdot_u B] A_{\alpha\beta} B^{\alpha\beta} \, , \tag{32}
$$

5. the spatial divergence of a spatial tensor

$$
[\mathrm{div}_u X]^{\alpha\ldots\beta} = \nabla(u)_\sigma X^{\sigma\alpha\ldots\beta} \, , \quad X = P(u) X \, . \tag{33}
$$

4 Kinematics of the observer's congruence

Let us consider now the splitting of the covariant derivative $\nabla_\beta u^\alpha$. This operation generates two spatial fields namely the acceleration vector field $a(u)$ and the kinematical tensor field $k(u)$defined as

$$
a(u) = P(u) \nabla_u u \, , \quad k(u) = -\nabla(u) u = \omega(u) - \theta(u) \ . \tag{34}
$$

where

$$
\begin{aligned}
[\omega(u)]_{\alpha\beta} &= -P(u)^\mu_\alpha P(u)^\nu_\beta \nabla_{[\mu} u_{\nu]} \ , \\
[\theta(u)]_{\alpha\beta} &= P(u)^\mu_\alpha P(u)^\nu_\beta \nabla_{(\mu} u_{\nu)} = \frac{1}{2} [\pounds(u)_u P(u)]_{\alpha\beta} \ ,
\end{aligned} \tag{35}
$$

are the components of tensor fields $\omega(u)$ and $\theta(u)$ having the meaning respectively of vorticity (whose sign depends on convention[1]) and expansion. From the above definitions, the tensor field $\nabla_\beta u^\alpha$ can be written as

[1] We have adopted the ∇-convention differently from a ;-convention also widely used.

$$\nabla_\beta u^\alpha = -a(u)^\alpha u_\beta - k(u)^\alpha{}_\beta\,. \tag{36}$$

The expansion tensor field $\theta(u)$ may itself be decomposed into its trace-free and pure trace parts

$$\theta(u) = \sigma(u) + \frac{1}{3}\Theta(u)P(u)\ , \tag{37}$$

where the trace-free tensor field $\sigma(u)$ ($\sigma(u)^\alpha{}_\alpha = 0$) is termed shear and the scalar

$$\Theta(u) = \nabla_\alpha u^\alpha \tag{38}$$

is termed volumetric (or isotropic) scalar expansion.

Define also the vorticity vector field $\omega(u) = 1/2\,\mathrm{curl}_u\, u$ as the spatial dual of the spatial rotation tensor, and given by

$$\omega(u)^\alpha = \frac{1}{2}\eta(u)^{\alpha\beta\gamma}\omega(u)_{\beta\gamma} = \frac{1}{2}\eta^{\sigma\alpha\beta\gamma}u_\sigma \nabla_\beta u_\gamma\,. \tag{39}$$

Although we use the same symbol for the vorticity tensor and the associated vector they can be easily distinguished by the context.

5 Adapted frames

Given a field of observers u, a frame $\{e_\alpha\}$ with $\alpha = 0,1,2,3$ (with dual ω^α) is termed adapted to u if $e_0 = u$ and e_a with $a = 1,2,3$ are orthogonal to u, namely $u\cdot e_a = 0$. From this it follows that $\omega^0 = -u^\flat$. In this section all indices denote components relative to the frame $\{e_\alpha\}$. The evolution of the frame vectors along the world lines of u is governed by the relations $\nabla_u e_\alpha = e_\sigma \Gamma^\sigma{}_{\alpha 0}$ and one can express the connection coefficients in terms of the kinematical quantities of the observer congruence u. The result is the following

$$\begin{array}{lll} \Gamma^a{}_{00} = a(u)^a, & \Gamma^0{}_{a0} = a(u)_a, & \Gamma^b{}_{a0} = C_{(\mathrm{fw})}{}^b{}_a, \\ \Gamma^b{}_{0a} = -k(u)^b{}_a, & \Gamma^0{}_{ba} = -k(u)_{ba}\,, & \end{array} \tag{40}$$

where the Fermi-Walker structure functions $C_{(\mathrm{fw})ba}$ are introduced so that

$$P(u)\nabla_u e_a = C_{(\mathrm{fw})}{}^b{}_a e_b\,. \tag{41}$$

They can also we written as

$$C_{(\mathrm{fw})}{}^b{}_a = C_{(\mathrm{lie})}{}^b{}_a - k(u)^b{}_a\,, \quad C_{(\mathrm{lie})}{}^b{}_a \equiv \omega^b(£(u)_u e_a)\,, \tag{42}$$

implying

$$P(u)£_u e_a = £(u)_u e_a = C_{(\mathrm{lie})}{}^b{}_a e_b\,. \tag{43}$$

Similarly, it is straightforward to express the structure functions in terms of kinematical quantities. In fact, from the definition

$$e_\alpha C^\alpha{}_{\beta\gamma} = [e_\beta, e_\gamma] = \nabla_{e_\beta} e_\gamma - \nabla_{e_\gamma} e_\beta \tag{44}$$

we have

$$\begin{aligned} e_\alpha C^\alpha{}_{0b} &= \nabla_u e_b - \nabla_{e_b} u = a(u)_b u + [C_{(\mathrm{fw})}{}^c{}_b + k(u)^c{}_b] e_c \\ &= a(u)_b u + C_{(\mathrm{lie})}{}^c{}_b e_c \,, \end{aligned} \tag{45}$$

so that $C^0{}_{0b} = a(u)_b$ and $C^c{}_{0b} = C_{(\mathrm{lie})}{}^c{}_b$. Similarly

$$e_\alpha C^\alpha{}_{bc} = \nabla_{e_b} e_c - \nabla_{e_c} e_b = 2\omega(u)_{bc} u + 2\Gamma^d{}_{[cb]} e_d \,, \tag{46}$$

so that $C^0{}_{bc} = 2\omega(u)_{bc}$ and $C^d{}_{bc} = 2\Gamma^d{}_{[cb]}$.

Finally, the structure functions satisfy the Jacobi identities which can also be given a $3+1$ form.

5.1 Spatial-Fermi-Walker and spatial-Lie temporal derivatives

We have introduced the Fermi-Walker structure functions $C_{(\mathrm{fw})}{}^b{}_a$,

$$P(u)\nabla_u e_a = C_{(\mathrm{fw})}{}^b{}_a e_b \,, \tag{47}$$

as well as the Lie structure functions $C_{(\mathrm{lie})}{}^b{}_a$ entering the projected Lie derivative along u (which we also termed as "spatial-Lie temporal derivative," see Eq. (20))

$$£(u)_u e_a = P(u) £_u e_a = C_{(\mathrm{lie})}{}^b{}_a e_b = C^b{}_{0a} e_b \,. \tag{48}$$

It is then useful to handle both these operations with a unified notation [1]

$$\nabla(u)_{(\mathrm{tem})} e_a = C_{(\mathrm{tem})}{}^b{}_a e_b \,, \quad \text{tem=fw, lie,} \tag{49}$$

where we define

$$\nabla(u)_{(\mathrm{fw})} e_a \equiv P(u)\nabla_u e_a, \qquad \nabla(u)_{(\mathrm{lie})} e_a \equiv P(u) £_u e_a = £(u)_u e_a. \tag{50}$$

Therefore if X is a vector field orthogonal to u, i.e. $X \cdot u = 0$, we have

$$\begin{aligned} \nabla(u)_{(\mathrm{tem})} X &= \nabla(u)_{(\mathrm{tem})} (X^a e_a) = \frac{dX^a}{d\tau_u} e_a + X^a C_{(\mathrm{tem})}{}^b{}_a e_b \\ &= \left(\frac{dX^b}{d\tau_u} + X^a C_{(\mathrm{tem})}{}^b{}_a \right) e_b = (\nabla(u)_{(\mathrm{tem})} X^b) e_b \,. \end{aligned} \tag{51}$$

The operation $\nabla(u)_{(\mathrm{fw})} = P(u)\nabla_u$ is termed "spatial-Fermi-Walker temporal derivative." It can be extended to non-spatial fields. If we apply this operation to the vector field u itself we have

$$\nabla(u)_{(\mathrm{fw})} u = P(u)\nabla_u u = a(u)\,. \tag{52}$$

Hence the temporal derivatives so defined through their action on purely spatial and purely temporal fields can now act on any spacetime field.

5.2 Frame components of the Riemann tensor

From the definition

$$e_\alpha R^\alpha{}_{\beta\gamma\delta} = [\nabla_{e_\gamma}, \nabla_{e_\delta}] e_\beta - C^\sigma{}_{\gamma\delta} \nabla_{e_\sigma} e_\beta\,, \tag{53}$$

we have

$$\begin{aligned} e_\alpha R^\alpha{}_{0b0} &= [\nabla_{e_b}, \nabla_u] u - C^\sigma{}_{b0} \nabla_{e_\sigma} u \\ &= \left\{ [\nabla(u)_b + a(u)_b] a(u)^c + \nabla(u)_{(\mathrm{fw})} k(u)^c{}_b - [k(u)^2]^c{}_b \right\} e_c\,, \end{aligned} \tag{54}$$

so that

$$R^c{}_{0b0} = [\nabla(u)_b + a(u)_b] a(u)^c + \nabla(u)_{(\mathrm{fw})} k(u)^c{}_b - [k(u)^2]^c{}_b\,, \tag{55}$$

where

$$\nabla(u)_{(\mathrm{fw})} k(u)^c{}_b = \nabla_u k(u)^c{}_b + C_{(\mathrm{fw})}{}^c{}_f k(u)^f{}_b - C_{(\mathrm{fw})}{}^f{}_b k(u)^c{}_f\,. \tag{56}$$

Similarly one obtains

$$R^0{}_{bcd} = -2[\nabla(u)_{[c} k(u)_{|b|d]} + \omega(u)_{cd} a(u)_b] \tag{57}$$

and

$$R^f{}_{bcd} = R_{(\mathrm{fw})}{}^f{}_{bcd} - 2k(u)_{b[c} k(u)^f{}_{d]}\,, \tag{58}$$

where

$$\begin{aligned} R_{(\mathrm{fw})}{}^f{}_{bcd} &= 2e_{[c}\left(\Gamma^f{}_{|b|d]}\right) + 2\Gamma^s{}_{b[c}\Gamma^f{}_{|s|d]} - C^s{}_{cd}\Gamma^f{}_{bs} \\ &\quad -2\omega(u)_{cd} C_{(\mathrm{fw})}{}^f{}_b\,. \end{aligned} \tag{59}$$

This tensor is termed "Fermi-Walker spatial Riemann tensor"[2]; it can be written in invariant form as follows

[2] A Lie spatial Riemann tensor can be defined similarly, replacing the Fermi-Walker structure functions $C_{(\mathrm{fw})}{}^f{}_b$ with the corresponding Lie structure functions $C_{(\mathrm{lie})}{}^f{}_b$ according to Eq. (42).

$$R_{(\mathrm{fw})}(u)(X,Y)Z = \{[\nabla(u)_X, \nabla(u)_Y]Z - \nabla(u)_{[X,Y]}\}Z \\ -2\omega(u)(X,Y)\nabla(u)_{(\mathrm{fw})}Z\,, \tag{60}$$

where X, Y and Z are spatial fields with respect to u and we note that

$$[X,Y] = P(u)[X,Y] - 2\omega^\flat(X,Y)\,u\,. \tag{61}$$

The Fermi-Walker spatial Riemann tensor has not all the symmetries of a three dimensional Riemann tensor. For instance it does not satisfy the Ricci identities. In fact we have

$$0 = R^f{}_{[bcd]} = R_{(\mathrm{fw})}{}^f{}_{[bcd]} - 2k(u)_{[bc}k(u)^f{}_{d]}\,, \tag{62}$$

and hence

$$R_{(\mathrm{fw})}{}^f{}_{[bcd]} = 2k(u)^f{}_{[b}\omega(u)_{cd]}\,. \tag{63}$$

From the latter one can construct a new Riemann tensor with all the necessary symmetries. Ferrarese [2] has shown that the symmetry-obeying Riemann tensor, denoted as $R_{(\mathrm{sym})}{}^{ab}{}_{cd}$, is related to the Fermi-Walker Riemann tensor (60) by

$$\begin{aligned} R_{(\mathrm{sym})}{}^{ab}{}_{cd} &= R_{(\mathrm{fw})}{}^{ab}{}_{cd} - 2\omega(u)^{ab}\omega(u)_{cd} - 4\theta(u)^{[a}{}_{[c}\omega(u)^{b]}{}_{d]} \\ &= R^{ab}{}_{cd} + 2k(u)^b{}_{[c}k(u)^a{}_{d]} - 2\omega(u)^{ab}\omega(u)_{cd} \\ &\quad -4\theta(u)^{[a}{}_{[c}\omega(u)^{b]}{}_{d]}\,. \end{aligned} \tag{64}$$

Together with the spatial symmetric Riemann tensor $R_{(\mathrm{sym})}{}^{ab}{}_{cd}$ we also introduce the spatial symmetric Ricci tensor, $R_{(\mathrm{sym})}{}^a{}_b = R_{(\mathrm{sym})}{}^{ca}{}_{cb}$ as well as the the associated scalar $R_{(\mathrm{sym})} = R_{(\mathrm{sym})}{}^a{}_a$.

6 Comparing families of observers

Let u and U be two unitary timelike vector fields. Define the relative spatial velocity of U with respect to u from the splitting relations

$$U = \gamma(U,u)[u + \nu(U,u)] = \gamma(U,u)[u + ||\nu(U,u)||\hat{\nu}(U,u)]\,, \tag{65}$$

and

$$u = \gamma(u,U)[U + \nu(u,U)] = \gamma(u,U)[U + ||\nu(u,U)||\hat{\nu}(u,U)]\,. \tag{66}$$

where $\hat{\nu}(U,u)$ is the unitary vector giving the direction of $\nu(U,u)$ in the rest frame of u. Both the spatial relative velocity vectors have the same magnitude

$$||\nu(U,u)|| = [\nu(U,u)_\alpha \nu(U,u)^\alpha]^{1/2} = [\nu(u,U)_\alpha \nu(u,U)^\alpha]^{1/2}.$$

The common gamma factor is related to that magnitude by

$$\gamma(U,u)=\gamma(u,U)=[1-||\nu(U,u)||^2]^{-1/2}=-U_\alpha u^\alpha \tag{67}$$

hence we recognize it as the relative Lorentz factor. It is convenient to abbreviate $\gamma(U,u)$ by γ and $||\nu(U,u||$ by ν when their meaning is clear from the context and there are no more than two observers involved.

Let us notice here that by substituting Eq. (65) into Eq. (66) we obtain the following relation

$$-\hat{\nu}(u,U)=\gamma[\hat{\nu}(U,u)+\nu u]\,, \tag{68}$$

which together with $U=\gamma[u+\nu\hat{\nu}(U,u)]$ yields the relative boost $B(U,u)$ from u to U, namely

$$\begin{aligned} B(U,u)u &= U=\gamma[u+\nu\hat{\nu}(U,u)]\\ B(U,u)\hat{\nu}(U,u) &= -\hat{\nu}(u,U)=\gamma[\hat{\nu}(U,u)+\nu u]\,. \end{aligned} \tag{69}$$

The inverse relations hold by interchanging U with u. The boost acts as the identity on the intersection of their local rest spaces $LRS_u\cap LRS_U$.

1. Maps between LRSs
   The spatial measurements of two observers in relative motion can be compared only relating their respective LRSs. Let U and u be two such observers and LRS_U and LRS_u their LRSs. There exists several maps between these LRSs; for example, by combining the projection operators $P(U)$ and $P(u)$ one can form the following "mixed projection" maps:

   a. $P(U,u)$ from the LRS_u into LRS_U, defined as

   $$P(U,u)=P(U)P(u):LRS_u\to LRS_U\,, \tag{70}$$

   with inverse: $P(U,u)^{-1}:LRS_U\to LRS_u$;

   b. $P(u,U)$ from the LRS_U into LRS_u, defined as

   $$P(u,U)=P(u)P(U):LRS_U\to LRS_u\,, \tag{71}$$

   with inverse: $P(u,U)^{-1}:LRS_u\to LRS_U$.

   Note that $P(U,u)\neq P(u,U)^{-1}$ as it follows from their representations

   $$\begin{aligned} P(U,u) &= P(u)+\gamma\nu U\otimes\hat{\nu}(U,u)\,,\\ P(U,u)^{-1} &= P(U)+\nu U\otimes\hat{\nu}(u,U)\,,\\ P(u,U) &= P(U)+\gamma\nu u\otimes\hat{\nu}(u,U)\,,\\ P(u,U)^{-1} &= P(u)+\nu u\otimes\hat{\nu}(U,u)\,. \end{aligned} \tag{72}$$

   One can then show that

$$P(U,u)\hat{\nu}(U,u) = -\gamma\hat{\nu}(u,U)\,,$$
$$P(u,U)^{-1}\hat{\nu}(U,u) = -\frac{1}{\gamma}\hat{\nu}(u,U)\,. \tag{73}$$

Note that

$$P(U,u) = P(U) \llcorner P(u) = P(U) \llcorner P(U,u) = P(U,u) \llcorner P(u)\ . \tag{74}$$

Moreover the following relations hold

$$P(U) = P(U,u) \llcorner P(U,u)^{-1}\ , \quad P(u) = P(U,u)^{-1} \llcorner P(U,u)\ . \tag{75}$$

2. The boost maps
Similarly to what we have done combining the projection maps, also the boost $B(U,u)$ induces an invertible map between the local rest spaces of the given observers defined as

$$B_{(\mathrm{lrs})}(U,u) \equiv P(U)B(U,u)P(u) : LRS_u \to LRS_U\ . \tag{76}$$

It acts as the identity on the intersection of their subspaces $LRS_U \cap LSR_u$. Being the boost an isometry, exchanging the role of U and u in (76) leads to the inverse boost

$$B_{(\mathrm{lrs})}(U,u)^{-1} \equiv B_{(\mathrm{lrs})}(u,U) : LRS_U \to LRS_u. \tag{77}$$

The representations of the boost and its inverse can be given in terms of the associated tensors

$$B_{(\mathrm{lrs})u}(U,u)\,, \quad B_{(\mathrm{lrs})U}(U,u)\,, \quad B_{(\mathrm{lrs})u}(u,U)\,, \quad B_{(\mathrm{lrs})U}(u,U), \tag{78}$$

defined by:

$$B_{(\mathrm{lrs})u}(U,u) = P(U,u)^{-1} \llcorner B_{(\mathrm{lrs})}(U,u)\ ,$$
$$B_{(\mathrm{lrs})U}(U,u) = B_{(\mathrm{lrs})}(U,u) \llcorner P(U,u)^{-1}, \tag{79}$$

with the corresponding expressions for the inverse boost obtained simply by exchanging the role of U and u and with

$$B_{(\mathrm{lrs})}(U,u) = B_{(\mathrm{lrs})U}(U,u) \llcorner P(U,u) = P(U,u) \llcorner B_{(\mathrm{lrs})u}(U,u)\ . \tag{80}$$

The explicit expression of $B_{(\mathrm{lrs})u}(U,u)$, for example, is given by

$$B_{(\mathrm{lrs})u}(U,u) = P(u) + \frac{1-\gamma}{\gamma}\hat{\nu}(U,u) \otimes \hat{\nu}(U,u). \tag{81}$$

This can be shown as follows. Let $X \in LRS_u$ then

$$
\begin{aligned}
B_{(\mathrm{lrs})u}(U,u)X &= P(U,u)^{-1}[B_{(\mathrm{lrs})}(U,u)X] \\
&= P(U,u)^{-1}[B_{(\mathrm{lrs})}(U,u)X^{||}\hat{\nu}(U,u)+X^{\perp}],
\end{aligned}
\tag{82}
$$

where $X^{||}=X\cdot\hat{\nu}(U,u)$ and $X^{\perp}=X-X^{||}\,\hat{\nu}(U,u)$, that is

$$
X=X^{||}\,\hat{\nu}(U,u)+X^{\perp}\,,
\tag{83}
$$

and we have used the fact that the boost reduces to the identity for vectors not belonging to the boost plane, as $X^{\perp}$. In this case the boost plane is spanned by the vectors u and $\hat{\nu}(U,u)$. Taking into account (69), namely

$$
B_{(\mathrm{lrs})}(U,u)\hat{\nu}(U,u)=-\hat{\nu}(u,U),
\tag{84}
$$

as well as the linearity of the boost map, we have

$$
\begin{aligned}
B_{(\mathrm{lrs})u}(U,u)X &= P(U,u)^{-1}[X-X^{||}\hat{\nu}(u,U)-X^{||}\hat{\nu}(U,u)] \\
&= X+\frac{1-\gamma}{\gamma}X^{||}\hat{\nu}(U,u),
\end{aligned}
\tag{85}
$$

where $P(U,u)^{-1}X=X$ because $X\in LRS_u$:

$$
\begin{aligned}
P(U,u)^{-1}X &= P(U,u)^{-1}P(u)X=P(U,u)^{-1}P(U)P(u)X \\
&= P(U,u)^{-1}P(U,u)X=P(u)X=X;
\end{aligned}
\tag{86}
$$

hence $P(U,u)^{-1}\hat{\nu}(U,u)=\hat{\nu}(U,u)$ because $\nu(U,u)$ belongs to LRS_u. Moreover, from (73), by exchanging the roles of U and u, we find

$$
\begin{aligned}
P(U,u)^{-1}\hat{\nu}(u,U) &= -\frac{1}{\gamma}\hat{\nu}(U,u)=\frac{1}{\gamma}B(u,U)\hat{\nu}(u,U) \\
&= \frac{1}{\gamma}B(U,u)^{-1}\hat{\nu}(u,U).
\end{aligned}
\tag{87}
$$

Therefore:

$$
\begin{aligned}
B_{(\mathrm{lrs})u}(U,u)X &= X-X^{||}\left(-\frac{1}{\gamma}+1\right)\hat{\nu}(U,u) \\
&= \left[P(u)-\frac{\gamma-1}{\gamma}\hat{\nu}(U,u)\otimes\hat{\nu}(U,u)^{\flat}\right]\llcorner X
\end{aligned}
\tag{88}
$$

which is equivalent to (81).
Similarly, for the inverse boost $B_{(\mathrm{lrs})}(u,U)$ one has

$$
\begin{aligned}
B_{(\mathrm{lrs})u}(u,U) &= P(u)-\frac{\gamma-1}{\gamma}\hat{\nu}(U,u)\otimes\hat{\nu}(U,u)^{\flat}\,, \\
B_{(\mathrm{lrs})U}(u,U) &= P(U)-\frac{\gamma-1}{\gamma}\hat{\nu}(u,U)\otimes\hat{\nu}(u,U)^{\flat}\,.
\end{aligned}
\tag{89}
$$

Thus, if S is a vector field such that $S \in LRS_U$, then its inverse boost is the vector belonging to LRS_u

$$B_{(\mathrm{lrs})}(u,U)S = [P(u) - \gamma(\gamma+1)^{-1}\nu(U,u)\otimes\nu(U,u)^\flat]\llcorner P(u,U)S\,. \tag{90}$$

7 Splitting of derivatives along a timelike curve

Consider a congruence of curves $\mathscr{C}_U$ with tangent vector field U and proper time τ_U as parameter. We know, at this stage, that the evolution along $\mathscr{C}_U$ of any tensor field can be specified by one of the following spacetime derivatives:

1. the absolute derivative along $\mathscr{C}_U$: $D/d\tau_U = \nabla_U$,
2. the Fermi-Walker derivative along $\mathscr{C}_U$: $D_{(\mathrm{fw},U)}/d\tau_U$,
3. the spacetime Lie derivative along $\mathscr{C}_U$: $\pounds_U$, for which we use also the notation $D_{(\mathrm{lie},U)}/d\tau_U = \pounds_U$.

The action of the Fermi-Walker and Lie derivatives on a vector field X is related to the absolute derivative as follows

$$\begin{aligned}
\frac{D_{(\mathrm{fw},U)}X}{d\tau_U} &= \nabla_U X + a(U)(U\cdot X) - U(a(U)\cdot X)\\
&= P(U)\nabla_U X - U\nabla_U(X\cdot U) + a(U)(X\cdot U)\,,\\
\frac{D_{(\mathrm{lie},U)}X}{d\tau_U} &= [U,X] = \nabla_U X + a(U)(U\cdot X) - k(U)\llcorner X\,,
\end{aligned} \tag{91}$$

where $k(U) = \omega(U) - \theta(U)$ is the kinematical tensor of the congruence $\mathscr{C}_U$ defined in (34). For $X = U$ we have

$$\frac{D_{(\mathrm{fw},U)}U}{d\tau_U} = 0\,, \qquad \frac{D_{(\mathrm{lie},U)}U}{d\tau_U} = 0\,, \tag{92}$$

whereas $DU/d\tau_U = a(U)$.

If X is spatial with respect to U, namely $X\cdot U = 0$, we have instead

$$\begin{aligned}
\frac{D_{(\mathrm{fw},U)}X}{d\tau_U} &= P(U)\nabla_U X\\
\frac{D_{(\mathrm{lie},U)}X}{d\tau_U} &= \nabla_U X - k(U)\llcorner X\\
&= P(U)\nabla_U X - U(a(U)\cdot X) - k(U)\llcorner X\,.
\end{aligned} \tag{93}$$

The projection orthogonal to U of $D_{(\mathrm{lie},U)}X/d\tau_U$ as in (93) gives

$$P(U)\frac{D_{(\mathrm{lie},U)}X}{d\tau_U} = \frac{D_{(\mathrm{fw},U)}}{d\tau_U}X - k(U)\llcorner X\,. \tag{94}$$

Let u be another family of observers whose world lines have as parameter the proper time τ_u. One can introduce on the congruence $\mathscr{C}_U$ whose unit tangent vector field can be written as

$$U = \gamma(U,u)[u + \nu(U,u)]\,, \tag{95}$$

two new parametrizations $\tau_{(U,u)}$ and $\ell_{(U,u)}$ as follows

$$\frac{d\tau_{(U,u)}}{d\tau_U} = \gamma(U,u)\,, \qquad \frac{d\ell_{(U,u)}}{d\tau_U} = \gamma(U,u)||\nu(U,u)||\,, \tag{96}$$

where $\tau_{(U,u)}$ corresponds to the proper times of the observers u when their curves are crossed by a given curve of $\mathscr{C}_U$ and $\ell_{(U,u)}$ corresponds to the proper length on $\mathscr{C}_U$. The projection orthogonal to u of the absolute derivative along U is expressed by

$$\begin{aligned} P(u)\frac{D}{d\tau_U} = P(u)\nabla_U &= \gamma[P(u)\nabla_u + P(u)\nabla_{\nu(U,u)}] \\ &= \gamma[P(u)\nabla_u + \nabla(u)_{\nu(U,u)}]\,. \end{aligned} \tag{97}$$

We note that in the above equation the derivative operation $P(u)\nabla_u$ is just what we have termed spatial-Fermi-Walker temporal derivative, i.e. $\nabla(u)_{(\mathrm{fw})}$, in (50). For a vector field X we can write then

$$\begin{aligned} P(u)\frac{D_{(\mathrm{fw},U)}X}{d\tau_U} &\equiv \frac{D_{(\mathrm{fw},U,u)}X}{d\tau_U} \\ &= P(u)\frac{DX}{d\tau_U} + P(u,U)a(U)(U\cdot X) \\ &\quad -\gamma\nu(U,u)(a(U)\cdot X)\,, \end{aligned} \tag{98}$$

$$\begin{aligned} P(u)\frac{D_{(\mathrm{lie},U)}X}{d\tau_U} &\equiv \frac{D_{(\mathrm{lie},U,u)}X}{d\tau_U} \\ &= P(u)\frac{DX}{d\tau_U} + P(u,U)a(U)(U\cdot X) \\ &\quad -P(u)[k(U)\llcorner X]\,. \end{aligned} \tag{99}$$

We shall now examine the projected absolute derivative in detail.

7.1 Projected absolute derivative

Consider the absolute derivative of u along U, namely $\nabla_U u$. Since u is unitary, then $u\cdot Du/d\tau_U = 0$ and we can write

$$\begin{aligned}\frac{Du}{d\tau_U} &= P(u)\frac{Du}{d\tau_U} = \gamma[P(u)\nabla_u u + P(u)\nabla_{\nu(U,u)}u]\\ &= \gamma[\nabla(u)_{(\mathrm{fw})}u + P(u)\nabla_{\nu(U,u)}u]\\ &= \gamma[a(u) + \omega(u)\times_u \nu(U,u) + \theta(u)\llcorner\nu(U,u)]\,. \end{aligned} \tag{100}$$

Let us denote the above quantity as (minus) Fermi-Walker gravitational force, namely

$$F^{(G)}_{(\mathrm{fw},U,u)} = -\frac{Du}{d\tau_U} = -\gamma[a(u) + \omega(u)\times_u \nu(U,u) + \theta(u)\llcorner\nu(U,u)]\,.$$

It should be stressed here that, although $F^{(G)}_{(\mathrm{fw},U,u)}$ is referred to as a gravitational force, it contains contributions by true gravity and by inertial forces.

Consider now the case of X orthogonal to u, i.e. $X\cdot u = 0$. The projection onto LRS_u of the absolute derivative of X along U gives

$$P(u)\frac{DX}{d\tau_U} = \gamma[P(u)\nabla_u X + \nabla(u)_{\nu(U,u)}X] \equiv \frac{D_{(\mathrm{fw},U,u)}X}{d\tau_U}\,. \tag{101}$$

This differential operator plays an important role since both Fermi-Walker and Lie derivatives along U can be expressed in terms of it. In terms of (adapted) frame components the above expression reads

$$P(u)\frac{DX}{d\tau_U} = \left\{\frac{dX^b}{d\tau_U} + \gamma\left[X^a\left(C_{(\mathrm{fw})}{}^b{}_a + \nu(U,u)^c\Gamma^b{}_{ac}\right)\right]\right\}e_b\,, \tag{102}$$

where we set $X = X^a e_a$. Introducing the relative standard time parametrization $\tau(U,u)$ defined in (96), we have

$$P(u)\frac{DX}{d\tau_{(U,u)}} = \frac{D_{(\mathrm{fw},U,u)}X}{d\tau_{(U,u)}} = \left(\frac{D_{(\mathrm{fw},U,u)}X}{d\tau_{(U,u)}}\right)^a e_a \tag{103}$$

or in components

$$\left(\frac{D_{(\mathrm{fw},U,u)}X}{d\tau_{(U,u)}}\right)^b = \frac{dX^b}{d\tau_{(U,u)}} + X^a\left(C_{(\mathrm{fw})}{}^b{}_a + \nu(U,u)^c\Gamma(u)^b{}_{ac}\right)\,. \tag{104}$$

A particular vector field which is orthogonal to u and is defined all along $\mathscr{C}_U$ is the field of relative velocities, $\nu(U,u)$. We introduce the acceleration of U relative to u by

$$a_{(\mathrm{fw},U,u)} = \frac{D_{(\mathrm{fw},U,u)}}{d\tau_{(U,u)}}\nu(U,u) = \gamma P(u)\frac{D}{d\tau_U}\nu(U,u)\,. \tag{105}$$

Considering instead the unit vector $\hat{\nu}(U,u)$, this quantity can be written as

$$a_{(\mathrm{fw},U,u)} = P(u)\frac{D}{d\tau_{(U,u)}}[\nu\hat{\nu}(U,u)]\,, \tag{106}$$

where $\nu = ||\nu(U,u)||$. Finally we have

$$a_{(\mathrm{fw},U,u)} = \hat{\nu}(U,u)\frac{d\nu}{d\tau_{(U,u)}} + \nu P(u)\frac{D}{d\tau_{(U,u)}}\hat{\nu}(U,u)\,. \tag{107}$$

It is therefore quite natural to denote the first term as a tangential Fermi-Walker acceleration, $a^{(T)}_{(\mathrm{fw},U,u)}$ of U relative to u and the second as centripetal Fermi-Walker acceleration $a^{(C)}_{(\mathrm{fw},U,u)}$ of U relative to u:

$$a_{(\mathrm{fw},U,u)} = a^{(T)}_{(\mathrm{fw},U,u)} + a^{(C)}_{(\mathrm{fw},U,u)}\,, \tag{108}$$

where

$$\begin{aligned} a^{(T)}_{(\mathrm{fw},U,u)} &= \hat{\nu}(U,u)\,\frac{d\nu}{d\tau_{(U,u)}}, \\ a^{(C)}_{(\mathrm{fw},U,u)} &= \nu\, P(u)\frac{D}{d\tau_{(U,u)}}\hat{\nu}(U,u) = \nu\,\frac{D_{(\mathrm{fw},U,u)}}{d\tau_{(U,u)}}\hat{\nu}(U,u). \end{aligned} \tag{109}$$

To generalize the classical mechanics notion of centripetal acceleration we need to convert the relative standard time parametrization into an analogous relative standard length parametrization[3]:

$$d\ell_{(U,u)} = \nu d\tau_{(U,u)}. \tag{110}$$

With this parametrization we have

$$\begin{aligned} a^{(C)}_{(\mathrm{fw},U,u)} &= \nu^2 P(u)\frac{D}{d\ell_{(U,u)}}\hat{\nu}(U,u) = \nu\,\frac{D_{(\mathrm{fw},U,u)}}{d\tau_{(U,u)}}\hat{\nu}(U,u) \\ &= \frac{\nu^2}{\mathscr{R}_{(\mathrm{fw},U,u)}}\hat{\eta}_{(\mathrm{fw},U,u)} = \nu^2 k_{(\mathrm{fw},U,u)}\hat{\eta}_{(\mathrm{fw},U,u)}\,, \end{aligned} \tag{111}$$

where $\hat{\eta}_{(\mathrm{fw},U,u)}$ is a unit spacelike vector orthogonal to $\hat{\nu}(U,u)$, $k_{(\mathrm{fw},U,u)}$ is the Fermi-Walker relative curvature and $\mathscr{R}_{(\mathrm{fw},U,u)}$ is the curvature radius of the curve such that

$$k_{(\mathrm{fw},U,u)}\hat{\eta}_{(\mathrm{fw},U,u)} = \frac{\hat{\eta}_{(\mathrm{fw},U,u)}}{\mathscr{R}_{(\mathrm{fw},U,u)}} = P(u)\frac{D}{d\ell_{(U,u)}}\hat{\nu}(U,u). \tag{112}$$

Clearly, if geometrically or physically motivated, one can replace the Spatial-Fermi-Walker temporal derivative with the Spatial-Lie temporal derivative defining the corresponding quantities. Doing this one really understands the power of the notation

[3] In fact the Euclidean space definition involves spatial orbits parameterized by the (spatial) curvilinear abscissa.

used. For example and for later use one can define Lie relative curvature of a curve and the associated curvature radius

$$k_{(\mathrm{lie},U,u)}\hat{\eta}_{(\mathrm{lie},U,u)} = \frac{\hat{\eta}_{(\mathrm{lie},U,u)}}{\mathscr{R}_{(\mathrm{lie},U,u)}} = \frac{D_{(\mathrm{lie},U,u)}}{d\ell_{(U,u)}}\hat{\nu}(U,u). \tag{113}$$

Difficult to think of other efficient relativistic generalizations of the classical concepts of inertial forces besides this one.

8 Preferred slicing is spacetimes admitting separable geodesics

After providing the general framework of spacetime splitting techniques let us briefly review now some recent results concerning the existence of preferred slicing in those spacetimes admitting separable geodesics (see [7] and references therein).

Let the coordinates x^α ($\alpha = 0\ldots 3$, with $x^0 = t$) be such that the geodesic equations are separable in the metric $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$. Using the Hamilton-Jacobi formalism we can write the tangent vector $U^\alpha = dx^\alpha(\lambda)/d\lambda$ to the affinely parametrized timelike geodesics as the gradient of the fundamental action function $S = S(x^\alpha,\lambda)$, $U_\alpha = \partial_\alpha S$, satisfying the Hamilton-Jacobi equation

$$-\frac{\partial S}{\partial\lambda} = H(x^\alpha,\partial_\alpha S)\,, \tag{114}$$

with λ an affine parameter for the integral curves of U and the Hamiltonian

$$H = \frac{1}{2}g^{\alpha\beta}\partial_\alpha S\partial_\beta S = -\frac{1}{2}\mu^2 = const\,, \tag{115}$$

the latter identity following from the normalization condition $U^\alpha U_\alpha = -\mu^2$ for timelike geodesics. Assume that S can be separated in its dependence on the variables x^α and λ, namely

$$S = \frac{1}{2}\mu^2\lambda + S_t(t) + S_1(x^1) + S_2(x^2) + S_3(x^3)\,. \tag{116}$$

Thus we have for the 1-form $U^\flat \equiv U_\alpha dx^\alpha = \partial_\alpha S dx^\alpha = d(S - \frac{1}{2}\mu^2\lambda)$, where here d stands for the spacetime differential only. Moreover, since in this case U is a gradient it is also necessarily vorticity-free: $dU^\flat = 0$, and there exists a distribution of constant action hypersurfaces $T \equiv -S + \frac{1}{2}\mu^2\lambda = const$ with

$$-dT = U_\alpha dx^\alpha\,, \tag{117}$$

such that U^α is the associated unit normal vector field. When one sets $\mu = 1$, then the time function T measures the proper time along the geodesics and the corresponding lapse function has the fixed value $N = 1$. For a stationary spacetime in which t is taken to be a Killing time coordinate, then $U_t = -E$ is a constant interpreted as a conserved energy, with $S_t(t) = -Et$, and the metric is independent of t. One then has

$$-dT = -Edt + U_a dx^a \,. \tag{118}$$

8.1 Static spherically symmetric spacetimes

Consider the case of static spherically symmetric spacetimes. The metric written in standard spherical-like coordinates is

$$ds^2 = -e^\nu dt^2 + e^\lambda dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{119}$$

where the functions ν and λ depend only on the radial coordinate. Then $L = U_\phi$ is an additional Killing constant associated with the conserved angular momentum so

$$U_\alpha dx^\alpha = -Edt + (\partial_r S_r)dr + (\partial_\theta S_\theta)d\theta + Ld\phi\,, \tag{120}$$

and the corresponding Hamilton-Jacobi equation

$$-e^{-\nu}E^2 + e^{-\lambda}(\partial_r S_r)^2 + \frac{1}{r^2}\left[(\partial_\theta S_\theta)^2 + \frac{L^2}{\sin^2\theta}\right] = -\mu^2 \tag{121}$$

can be easily separated in its dependence on the coordinates by setting the square bracket expression to a separation constant $\mathscr{K}$, leading to

$$\frac{dS_r}{dr} = \varepsilon_r e^{\lambda/2}\sqrt{E^2 e^{-\nu} - \frac{\mathscr{K} + \mu^2 r^2}{r^2}}\,, \qquad \frac{dS_\theta}{d\theta} = \varepsilon_\theta\sqrt{\mathscr{K} - \frac{L^2}{\sin^2\theta}}\,, \tag{122}$$

where $|\varepsilon_r| = 1 = |\varepsilon_\theta|$.

As stated above, we can set $\mu = 1$ to characterize a new foliation by a new temporal coordinate T measuring proper time along the orthogonal geodesics. We are left to specify E, L, and $\mathscr{K}$ to obtain a specific family of timelike geodesics covering the spacetime. The simplest choice would be a spherically symmetric 4-velocity field involving only radial motion of the geodesics relative to the original coordinates. We can achieve this in two steps. First we can require that this family of geodesics be tangent to the equatorial plane $\theta = \pi/2$, which requires $\mathscr{K} = L^2$ to make $U_\theta = 0$, resulting in

$$U_r^2 = e^{\lambda-\nu}\left[E^2 - e^\nu\left(1 + \frac{L^2}{r^2}\right)\right]\,. \tag{123}$$

We then impose the radial condition $L = 0$, so that

$$U_r = \varepsilon_r e^{(\lambda-\nu)/2}\sqrt{E^2 - e^\nu}\,, \tag{124}$$

leaving finally the choice of the energy constant E. For spatially asymptotically flat spacetimes where $e^{\nu} < 1$ approaches 1 as $r \to \infty$, to have a choice which works even at spatial infinity, we must have $E \geq 1$, in which case the value may be interpreted as the energy of the radially moving geodesics at spatial infinity. Of course one could choose $E < 1$ but this would limit the slicing to the interior of a cylinder in spacetime inside the radial turning point of the geodesic motion.

The new time differential is then

$$dT = Edt - U_r\, dr\,. \tag{125}$$

A new global coordinate system for static spacetimes is given by $(X^{\alpha}) = (T, R, \theta, \phi)$ with $R = r$ and θ and ϕ unchanged and $T = Et + f(r)$ given by integrating the differential equation $f'(r) = -U_r$. This leads to

$$\partial_T = E^{-1}\partial_t\,, \quad \partial_R = \partial_r + \frac{U_r}{E}\partial_t\,, \tag{126}$$

and the transformed metric is

$$ds^2 = -dT^2 + \gamma_{ab}(dX^a + N^a dT)(dX^b + N^b dT)\,, \tag{127}$$

with unit lapse function and the shift vector field aligned with the new radial direction, i.e.,

$$N^a = -\delta^a_R U^r = -\delta^a_R e^{-\lambda} U_r = -\delta^a_R \varepsilon_r e^{-(\lambda+\nu)/2}\sqrt{E^2 - e^{\nu}}\,. \tag{128}$$

The 3-metric induced on the $T = const$ hypersurfaces is then given by

$${}^{(3)}ds^2 = \frac{e^{\lambda+\nu}}{E^2}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{129}$$

In the case of vacuum as well as in the presence of a nonzero cosmological constant one has $\lambda + \nu = 0$, so that the induced metric is then

$${}^{(3)}ds^2 = \frac{dr^2}{E^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{130}$$

whose only nonvanishing component of the spatial Riemann curvature tensor and the spatial curvature scalar are

$${}^{(3)}R^{\theta\phi}{}_{\theta\phi} = \frac{1 - E^2}{r^2} = \frac{1}{2}{}^{(3)}R\,, \tag{131}$$

with positive or negative curvature respectively for $0 < E < 1$ (bound geodesics) or $E > 1$ (unbound geodesics). The choice $E = 1$ leads to a flat 3-geometry. The additional sign choice $\varepsilon_r = -1$ corresponds to the radially infalling geodesics which start at rest at spatial infinity. This is the case for the Schwarzschild spacetime where the Painlevé-Gullstrand coordinates were originally found [8, 9].

9 Discussion

The results summarized above have been developed over a period of about 20 years (starting from the 90s), initially motivated by the necessity to correctly define inertial forces in general relativity. As a consequence, the whole "measurement process" was reformulated, paying much attention to involve only quantities with a clear geometrical and physical meaning.

Actually, the relativistic generalization of well known classical quantities necessitated the introduction of the so called observer's viewpoint and, formally, the systematic use of "1+3" spacetime splitting techniques. Relative Frenet-Serret frames, for instance, were perhaps the most suited tools to explain how inertial forces could enter the general relativistic dynamics of test particles, in full similarity with the classical situation.

A lot of progress in this field (which was not at all a newborn field) was possible because of the international competition started in analyzing explicit applications of formalism to test particle motion in black hole spacetimes. The original enthusiasm was swept away when a satisfactory understanding of the problem was obtained. Nevertheless, many aspects of the formalism developed in this field may be exported in different contexts and hence one should wait for another wave of splitting formalism when new applications will be taken into account.

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