Quasinormal modes from a naked singularity

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Abstract What should be the quasinormal modes associated with a spacetime that contains a naked singularity instead of a black hole? In the present work we address this problem by studying the scattering of scalar fields on a curved background described by a Reissner-Nordström spacetime with $q > m$. We show that there is a qualitative difference between cases with $1 < q^2/m^2 \lesssim 9/8$ and cases with $q^2/m^2 \gtrsim 9/8$. We discuss the necessary conditions for the well-posedness of the problem, and present results for the low l and large l limit.

1 Introduction

The naked Reissner-Nordström (R-N) singularity is a classical general relativistic solution in electrovacuum. The solution is expected to have a very limited meaning, due to the fact that such singularities cannot be created neither by a gravitational collapse, nor by dropping a charge into the black hole. (By weak cosmic censorship conjecture general naked singularities should be prohibited in general theory of relativity, although there are indications that by including quantum effects the violations of the conjecture could be considered [1].) Moreover, a naked singularity created from some exotic initial data conditions should become quickly neutralized (classically, or via quantum pair production). Some results also indicate that if one considers electro-gravitational perturbations the R-N naked singularity becomes linearly unstable [2]. However it was discovered that the scalar field scattering problem

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on such a singular background can be still well defined [3, 4, 5, 6, 7, 8], since the waves remain regular at the origin. Despite of this nice property of the scattering problem, the spacetime is non-globally hyperbolic and the time evolution of the fields is not unique [9, 10]. This means one has to specify additional boundary condition at the singularity to obtain a fully unique time evolution. Another way of seeing the problem is through the language of operators: one can understand the spatial part of the wave operator to be a positive symmetric operator acting on a L^2 Hilbert space, and then obtain the scalar field dynamics through a suitable positive self-adjoint extension of such a symmetric operator [3, 4]. (One "preferred" way in which such a self-adjoint extension can be always realized is through the so called Friedrich's extension [3], which will also be the case of this paper.) Anyway, after uniquely specifying the dynamics, one should be able to characterize the scattering by a set of characteristic oscillations, the quasi-normal modes.

Low damped quasi-normal modes are in general used as a possible source of information about potential astrophysical objects (such as neutron stars, black holes), the highly damped modes are potentially interesting from the point of view of quantum gravity. Since a lot of work was devoted to the problem of quasinormal modes of the Reissner-Nordström black hole, it might be interesting to observe what happens if one transits from the R-N black hole case to the R-N naked singularity case (with a reflective boundary condition). Information about "what happens" shows how many features of the quasi-normal modes of the black hole spacetimes are specific to the black holes themselves and what features survive much more general conditions. Thus, briefly, we hope that despite of the fact that most likely the R-N naked singularity model does not correspond to a realistic physical situation, there are still many interesting things one can learn from such a model. One of them is a question that we want to answer in the present paper, in particular what will be the behaviour of the *low* damped quasinormal modes when departing from the R-N black hole to the R-N naked singularity.

2 The time evolution problem for a scalar field in the R-N naked singularity

In this section we will follow the standard analysis of the scalar field evolution in a curved background. (As an example of such an analysis see the treatment of Schwarzschild black hole perturbations in [11, 12]. For a review that presents also such techniques see for example [13].) Take the Klein-Gordon equation for the complex (charged) scalar field:

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\, g^{\mu\nu}\partial_\nu\Psi) = 0, \tag{1}$$

with the metric line element given as

$$g_{\mu\nu}dx^{\mu}dx^{\nu} = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2. \tag{2}$$

For the Reissner-Nordström (R-N) singularity the function $f(r)$ is in Planck units given as:

$$f(r) = 1 - \frac{2m}{r} + \frac{q^2}{r^2}, \quad (q^2 > m^2). \tag{3}$$

Take the decomposition of the field into the spherical harmonics

$$\Psi(t,r,\theta,\phi) = \sum_{l,m} \psi_l(t,r) Y_{ml}(\theta,\phi). \tag{4}$$

After we separate the variables we obtain the following reduced equation

$$\frac{d^2\psi_l(t,r)}{dt^2} = \frac{f(r)}{r^2}\frac{d}{dr}\left[r^2 f(r)\frac{d\psi_l(t,r)}{dr}\right] - \frac{l(l+1)f(r)}{r^2}\psi_l(t,r). \tag{5}$$

The unique solution of the given initial data problem is obtained if the condition of boundedness of the Green's function leads to a *unique* way to construct Green's function from the two linerly independent solutions U_{l1}, U_{l2} of the homogeneous equation associated to (5). Unfortunately for the case of R-N naked singularity both of the linearly independent solutions U_{l1}, U_{l2} are regular at 0 and the problem is underdetermined.

Is there any intuitive physical condition that we can further impose on the fields, that will uniquely select the appropriate Green's function? At least to get the geometrical optics continuous extension of the black hole case one can impose the condition that nothing falls in or out of the singularity. This means there is neither absorption nor superradiation in the scattering and the S-matrix of the K-G field is a unitary operator.

Further in the text we will employ the field vanishing condition at 0. (This boundary condition at the singularity corresponds to what is known as Friedrich's extension of a symmetric operator.) Thus the quasinormal modes will relate to the scattering problem following from the time evolution determined by the boundary condition $\psi(0,t) = 0$ [14].

3 The scalar wave scattering on a naked singularity

Using ϕ_l defined as $\phi_l(r,t) = r\psi_l(r,t)$ and x the tortoise coordinate given by the condition:

$$\frac{dr}{dx} = f(r), \tag{6}$$

one can rewrite the equation (5) into the following form

$$\frac{\partial^2\phi_l(x,t)}{\partial t^2} - \frac{\partial^2\phi_l(x,t)}{\partial x^2} = V(m,q,l,x)\phi_l(x,t), \tag{7}$$

with

$$V(m,q,l,x) = \left[\frac{l(l+1)}{r^2(x)} + \frac{2m}{r^3(x)} - \frac{2q^2}{r^4(x)}\right] f\big(r(x)\big). \tag{8}$$

And, for the normal modes $e^{-i\omega t}\phi_l(r)$, we can write

$$\frac{\partial^2 \phi_l(x)}{\partial x^2} + \left[\omega^2 - V(m,q,l,x)\right]\phi_l(x) = 0. \tag{9}$$

If $q > m$, we can see that $f(r)$ given by eq. (3) has no zeros for real arguments, but eq. (6) can be analytically integrated.

The potential (8) has for the ratio q^2/m^2 less than approximately $9/8$ and the relevant x (in the naked singularity case the domain of x is constrained) 3 extrema, one smaller "outer" maximum, one dominant "inner" maximum and minimum in the potential valley between them. (For $r \to 0$ the function $V(r) \to -\infty$.) For q^2/m^2 more than approximately $9/8$ the potential has only one maximum (thus only one peak). These features of the potential (8) can be seen in the figure 1.

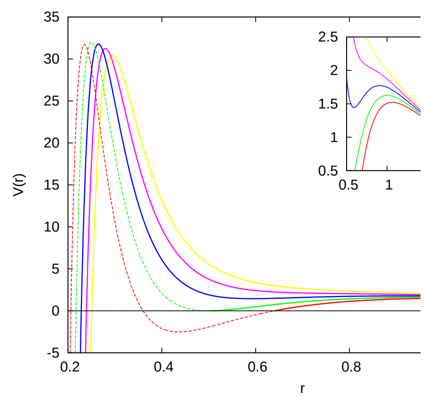

Fig. 1 Potential $V(r)$ given by eq. (8) with $l = 2$, $m = 0.5$ for $q = 0.48, 0.5, 0.52, 0.54$ and 0.56. (The curves from left to right correspond to the increase of charge.) Note that the dashed part of the potential (for $q = 0.48$ and 0.5) is inside the black hole horizon.

4 The naked singularity for the small wave mode numbers - numerical results for the frequencies

Our objective in this section is to solve eq. (7) with potential (8) numerically, in the case where $q > m$ as described in the last section. To do this, we rewrite eq. (7) in terms of the light-cone variables $u = t - x$ and $v = t + x$, where x corresponds to the tortoise coordinate (6), as

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} = -4\frac{\partial^2 \phi}{\partial u \partial v} = V(r)\phi\,, \tag{10}$$

that can be integrated with the boundary conditions

$$\phi(r=0,t) = \phi(u, v = u + 2x_0) = 0\,, \tag{11}$$

$$\phi(u=0,v) = e^{-\frac{(v-v_c)^2}{2\sigma^2}}\,, \tag{12}$$

where condition (11) is a necessary condition on the field ϕ near the origin (see the discussion on fig. 2 below) and condition (12) defines an "arbitrary" relevant initial signal to be propagated. We use the algorithm

$$\phi_N = \phi_W + \phi_E - \phi_S - \frac{\phi_W + \phi_E}{8} V \Delta_v \Delta_u\,, \tag{13}$$

where Δ_u and Δ_v are the integration steps in u and v, respectively. Note that here V is the potential (8) evaluated at the same r coordinate as ϕ_S (and ϕ_N).

As we can see in fig. 2, the boundary conditions (11) and (12) ensure the necessary conditions on the fields ϕ and ψ near the center. As we discussed previously in section 2, the physically correct boundary condition for ψ is $\psi(0,t) = 0$. From this we must have for $\phi(r,t) = r\psi(r,t)$ that $\phi(0,t) = 0$ and $\phi'(0,t) = 0$.

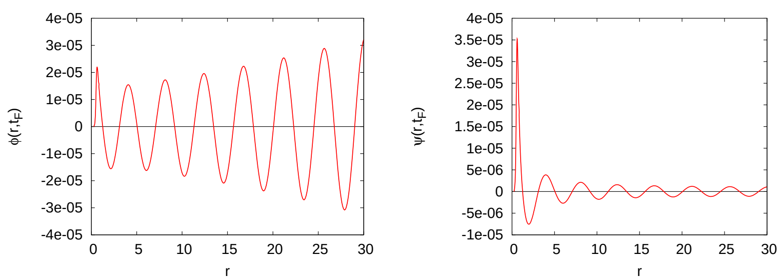

Fig. 2 Left: Behavior of ϕ with $l = 2$ as a function of r near the center $r = 0$ for a late time $t_F = 350$, shown here in order to exemplify the effect of conditions (11) and (12) in the numerical integration, for a spacetime with $q = 0.5$ and $q = 0.52$. Right: The same as in the left plot, but this time for the function $\psi = \phi/r$.

In the left plot of fig. 3 we present some typical time evolutions of ϕ, for a $l = 2$ and different q/m ratios. In the right plot we present the obtained frequencies of the QNMs (fundamental mode) in the $\omega_R \times \omega_I$ plane. We can see a discontinuity in the frequencies as $q/m \to 1$, as was expected from the discussion of the potential $V(r)$ (see fig. 1). We also point here that we see no significant changes, but rather a smooth behavior as $q^2/m^2 \to 9/8$ ($q/m \to 1.06$ in the plot). But we see a point of inflection in ω_R at $q/m \approx 1.16$, for which we did not find an analytical explanation.

Finally, in fig. 4 we explore how the frequencies of the QNMs change with l. As usual in black hole scattering problems, we see that the oscillation frequency ω_R increases with l. But the qualitative behaviour of ω_I changes significantly with q/m. In the upper plots ($q^2/m^2 \lesssim 9/8$), $|\omega_I|$ decreases with l, that is, the damping

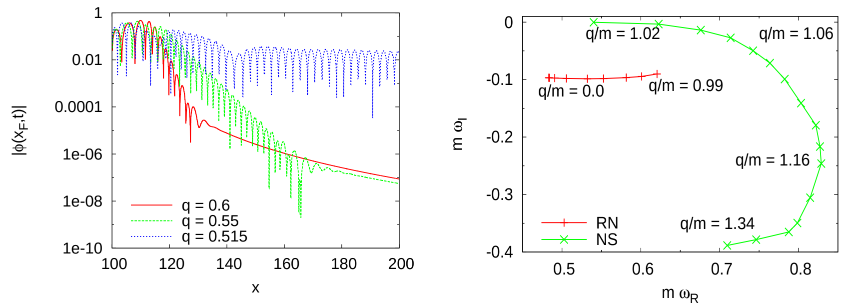

Fig. 3 Left: $\phi(x_F,t)$ with $l=2$ at $x_F=100$ for a spacetime with $m=0.5$ and different values of $q>m$. Right: Frequencies of the fundamental mode with $l=2$ in the $\omega_R \times \omega_I$ plane, parametrized by the q/m ration.

time is longer. In the lower plots ($q^2/m^2 \gtrsim 9/8$), we have the opposite tendency. This behaviour is connected to the potential $V(r)$. It might be also interesting to mention that in case $q^2/m^2 \gtrsim 9/8$, there is a qualitative similarity in the behaviour of the imaginary part of the frequencies as a function of l, between the case when l is small and the l large limit.

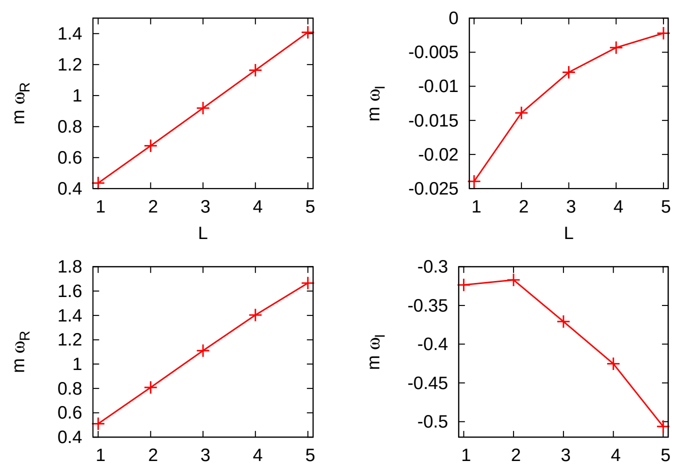

Fig. 4 Above: frequencies of the fundamental mode as a function of l for $m=0.5$ and $q=0.52$ ($q^2/m^2 < 9/8$). Below: same as above, but this time for $m=0.5$ and $q=0.6$ ($q^2/m^2 > 9/8$).

5 Conclusions

In this paper we analysed the problem of scalar field scattering on a R-N naked singularity background from the point of view of quasi-normal modes. The evolution on the R-N naked singularity is non-unique unless one specifies additional boundary condition representing a "hair" of the singularity. The quasi-normal modes then carry information about the "hair". We applied a particular boundary condition, that nothing comes out, or in from the singularity and analysed analytically, as well as numerically, the characteristic oscillations of the scalar field perturbations (low damped quasi-normal modes). In [14] we also analysed the eikonal $l \gg 1$ case via the analytical approach confirming the intuition obtained through the massless particle viewpoint, and showed that an approach based on analytical approximations can be useful also for the small l wave mode numbers. For the small l-s we calculated the frequencies numerically via the characteristic integration method.

The basic results can be summarized as follows: for the large l there is a continuous transition in the low damped QNM modes between the R-N black hole and the R-N naked singularity (see [14]). However, when the ratio q^2/m^2 becomes larger than approximately $9/8$ then the picture becomes significantly different and the low damped modes do *not* exist for large l-s. (This is a very different picture from the BH based intuition.) For the small l numbers the modes face a discontinuous transition when transiting from the black hole to the naked singularity. Furthermore, the l dependence $|\omega_I|$ (for small l) changes as q^2/m^2 becomes larger than approximately $9/8$: $|\omega_I|$ decreases for $q^2/m^2 \lesssim 9/8$ and increases for $q^2/m^2 \gtrsim 9/8$. It might be interesting to notice that for $q^2/m^2 \gtrsim 9/8$ the increase of $|\omega_I|$ as a function of l (for small l-s) matches the behaviour of $|\omega_I|$ for large l-s. In the case of large l-s and $q^2/m^2 \gtrsim 9/8$ we have shown that $|\omega_I|$ of the fundamental mode grows at least cubically with l and thus, as we already mentioned, the low damped modes do not exist [14].

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