Scalar Fields on anti-de Sitter Background

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Abstract The study of scalar fields coupled to gravity when there is a negative cosmological constant gives important insight on the possible instability of anti-de Sitter spacetime. In this short paper we consider the question how different the scalar field evolution is when the background is a fixed AdS metric. It is known that self-interacting massive real scalar fields on flat Minkowski background can form long living oscillating localized objects, named oscillons. In the flat background case these objects radiate energy extremely slowly, in a rate which is exponentially suppressed in terms of the central amplitude. However, on AdS background there are localized exactly time-periodic non-radiating solutions.

1 Introduction

In a recent influential paper Bizoń and A. Rostworowski [1] studied a real scalar field coupled to gravity when there is a negative cosmological constant. In this case the geometry approaches asymptotically the anti-de Sitter spacetime. They considered the time evolution of a spherically symmetric massless scalar field, and observed that the energy is continuously shifted to small wavelength high frequency modes. This phenomenon is generally called weak turbulence in the literature. The shift of energy to high frequency modes continues until a black hole forms at the symmetry center. In this short paper we consider the question what changes when

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the background is not dynamical but a fixed AdS spacetime. For a Klein-Gordon field on a fixed background the field equations are linear and consequently there is no weak turbulence. For this reason we consider self-interacting scalar fields.

We consider $3+1$ dimensional anti-de Sitter spacetime in the conformal coordinate system

$$ds^2 = \frac{1}{k^2 \cos^2 x} \left(-d\tau^2 + dx^2 + \sin^2 x \, d\Omega^2 \right) , \tag{1}$$

where $d\Omega$ is the metric of a unit two-sphere. In these coordinates $x = 0$ corresponds to the center of symmetry, and $x = \pi/2$ to infinity. All timelike geodesics emanating from a point meet again at another point. A light ray can travel to infinity and back in finite coordinate time, if we assume that infinity acts as a mirror for null rays. The behavior of geodesics indicates that the AdS background corresponds to an effective attractive force.

2 Evolution of a scalar field on AdS background

A spherically symmetric self-interacting scalar field on $3+1$ dimensional AdS background evolves according to the field equation

$$-\phi_{,\tau\tau} + \phi_{,xx} + \frac{4}{\sin(2x)} \phi_{,x} = \frac{U'(\phi)}{k^2 \cos^2 x} , \tag{2}$$

where k is related to the cosmological constant by $\Lambda = -3k^2$, and $U(\phi)$ is the potential describing the self-interaction of the scalar field. In order to show that weak turbulence is likely to occur even in this simpler system, let us consider a specific example. We choose $k = 1$ and the scalar potential as $U(\phi) = \frac{1}{2}\phi^2 - \frac{1}{4}\phi^4 + \frac{1}{6}\phi^6$. For initial data we take a finite width spherically symmetric shell, for which

$$\phi = c \exp \frac{b^2 d}{(x-a)^2 - b^2} \tag{3}$$

for $|x - a| < b$ and $\phi = 0$ otherwise. For the concrete example that we present here we have chosen the constants as $a = 0.4$, $b = 0.2$, $c = 100$ and $d = 4$. At the beginning the shell separates into ingoing and outgoing shells. The ingoing shell approaches the center and then becomes outgoing. Later both shells get reflected back from infinity in a finite time. It takes approximately $\pi/2$ time interval in the coordinate time τ for a shell to go from the center to infinity, or to come back, quite similarly to how null geodesics behave. On the top panel of Fig. 1 we show the time evolution of the scalar field ϕ at the center for the first few reflections. The bottom panel shows the evolution of the central value of the energy density,

$$\varepsilon = k^2 \cos x \left(\frac{1}{2} (\phi_{,\tau})^2 + \frac{1}{2} (\phi_{,x})^2 + \frac{U(\phi)}{k^2 \cos^2 x} \right) . \tag{4}$$

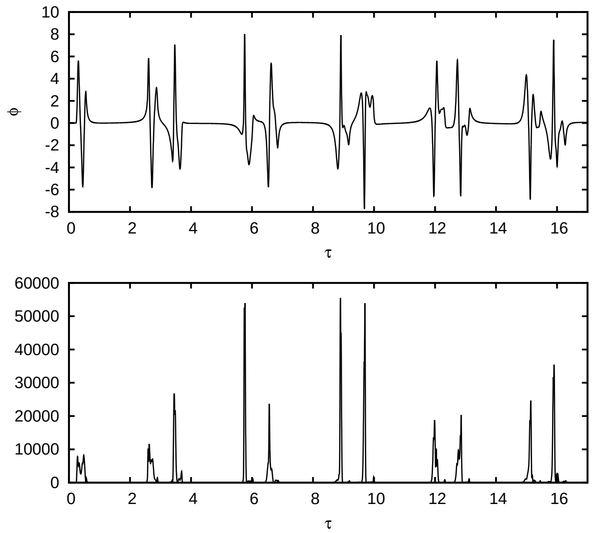

Fig. 1 Time dependence of the scalar field and its energy density at the center.

On the energy density plot there are peaks when the shells come to the center. Looking at a much longer time interval, it can be seen that the amplitude of the peaks increases very quickly. On Fig. 2 we show the central energy density for a longer time interval, on a logarithmic plot. The energy density increases about five magnitudes by the time the shells are reflected about one hundred times. It seems very

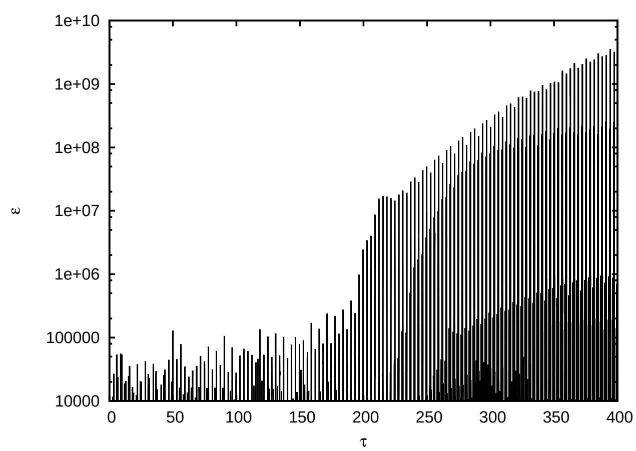

Fig. 2 Time dependence of the energy density at the center.

likely that there is week turbulence in this system. It would take further numerical work to study how the energy density increases for other types of initial data and for different choices of the scalar potential. For massless self-interacting fields, such as that with a potential $U(\phi)=\phi^4$, we could not observe a significant increase in the central energy density. For the massive case there is density increase, but we could not see any simple scale invariance property of the time evolution depending on the amplitude of the initial data, which has been, however, observed for the self-gravitating massless Klein-Gordon case in [1].

3 Periodic solutions

In a recent paper of Dias, Horowitz and Santos [2], vacuum spacetimes have been considered when there is a negative cosmological constant. Using perturbation theory the existence of resonant modes has been shown, which indicates weak turbulence. It has been also claimed that the nonlinear generalization of a single perturbative mode is a localized periodic vacuum solution, which is generally called geon in the literature [3, 4]. Geons are not spherically symmetric, but if one includes a scalar field in the system, then spherically symmetric localized periodic solutions are expected to exist, which are called oscillatons in the asymptotically flat case [5, 6]. Oscillatons are similar to boson stars [7], but for boson stars the scalar field is complex, and the metric is static. Similar periodic localized solutions already exist for scalar fields on a fixed AdS background. On flat Minkowski background those objects are known as breathers or oscillons [8].

For the case of a massive or massless Klein-Gordon field on AdS background the periodic solutions are explicitly known. In this case the scalar potential is $U(\phi)=\frac{1}{2}m^2\phi^2$, and there is a family of breather solutions [9] labeled by a non-negative integer n, which gives the number of the nodes of the solution

$$\phi^{(n)}=\cos[(\mu+2n)\tau]\,(\cos x)^{\mu}\,P_n^{(1/2,\mu-3/2)}\,(\cos(2x))\,, \tag{5}$$

where

$$\mu=\frac{3}{2}+\sqrt{\frac{9}{4}+\frac{m^2}{k^2}}\ , \tag{6}$$

and $P_n^{(a,b)}(x)$ denotes the Jacobi polynomial. All finite energy solutions can be expressed as sums of $\phi^{(n)}$ with appropriate phases.

Self-interacting scalar fields with any potential $U(\phi)$ also admit periodic localized solutions on AdS background. We can look for solutions oscillating with frequency ω by Fourier decomposing the scalar field in the form

$$\phi=\sum_{n=0}^{N}\phi_n\cos(n\,\omega\,\tau)\ , \tag{7}$$

where ϕ_n are functions of x and the system is truncated at order N. We have solved the resulting system of ordinary differential equations using the spectral code Kadath developed by Philippe Grandclément [10]. On Fig. 3 the Fourier modes of an example configuration are given for the standard ϕ^4 potential $U(\phi) = \phi^2(\phi-2)^2/8$, when $k = 1$ and $\omega = 3.6$. Unlike in the linear Klein-Gordon case, the frequency ω

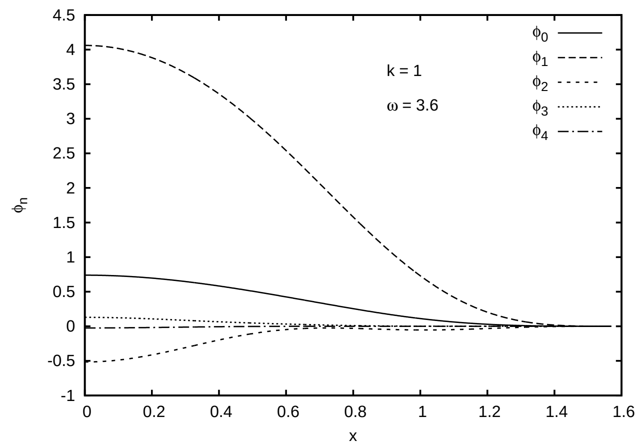

Fig. 3 The first few Fourier modes of a periodic localized solution in case of a ϕ^4 potential.

changes when the oscillation amplitude grows. In the limit of small oscillations ω tends to the Klein-Gordon value $\omega_0 = \mu = (3+\sqrt{13})/2 \approx 3.30278$ given by (6).

A small-amplitude expansion procedure has been successfully applied in the past for oscillons [8], asymptotically flat oscillatons [11], and also for oscillatons when there is a small positive cosmological constant [12]. In this paper we present the small-amplitude expansion of oscillons on a fixed AdS background when $|\Lambda|$ is small. We use Schwarzschild area coordinates,

$$ds^2 = -(1+k^2r^2)dt^2 + \frac{dr^2}{1+k^2r^2} + r^2 d\Omega^2 \; . \tag{8}$$

Then the field equation takes the form

$$-\frac{1}{1+k^2r^2}\phi_{,tt} + (1+k^2r^2)\phi_{,rr} + \left[\frac{D-1}{r} + (D+1)k^2 r\right]\phi_{,r} = U'(\phi) \; . \tag{9}$$

We describe the scalar potential by its expansion coefficients g_k,

$$U(\phi) = m^2\left(\frac{1}{2}\phi^2 + \frac{g_2}{3}\phi^3 + \frac{g_3}{4}\phi^4 + \ldots\right), \tag{10}$$

where m is the mass of the scalar field. We expand the scalar field in powers of a small parameter ε

$$\phi = \varepsilon\phi_1 + \varepsilon^2\phi_2 + \varepsilon^3\phi_3 + \ldots \ . \tag{11}$$

Since on Minkowski background the size of small amplitude oscillons scales as $1/\varepsilon$, we use a rescaled radial coordinate $\rho = \varepsilon m r$. This makes spatial derivatives one order smaller. We also define a new time coordinate by $\tau = m\omega t$. The ε dependence of the oscillation frequency ω is represented by

$$\omega^2 = 1 + \omega_2\varepsilon^2 + \omega_4\varepsilon^4 + \ldots \ , \tag{12}$$

where ω_k are constants. We also introduce a rescaled cosmological parameter κ by $k = \varepsilon^2 m\kappa$. This ensures that the oscillon size remains small compared to the curvature scale in the ε tends to zero limit. Substituting the expansion (11) into the field equation (9), to leading ε order we obtain that $\phi_1 = p_1\cos\tau$, where p_1 depends only on ρ. The radial dependence of p_1 will be determined by the absence of secular terms in ϕ_3, yielding

$$p_{1,\rho\rho} + \frac{D-1}{\rho}p_{1,\rho} + \left(\omega_2 - \rho^2\kappa^2\right)p_1 + \lambda p_1^3 = 0 \ , \tag{13}$$

where $\lambda = \frac{5}{6}g_2^2 - \frac{3}{4}g_3$. This gives the spatial profile of the oscillon to leading order. For Minkowski background $\kappa = 0$, and localized solutions can exist only if λ is positive. If $\lambda > 0$ we may consider the potential as an "attractive potential". In this case the potential is more flat near its minimum than the same mass harmonic potential, and the oscillation period becomes longer. For anti de Sitter background we can rescale ρ, and consequently ε in $\rho = \varepsilon m r$, in order to set

$$\kappa = 1 \ , \tag{14}$$

which we assume from now on.

If $\lambda = 0$, then (13) is linear, and there are localized solutions only if $\omega_2 = 3 + 4n$, for $n \geq 0$ integer. The solutions can be written in terms of generalized Laguerre polynomials,

$$p_1^{(n)} = \exp\left(-\frac{\rho^2}{2}\right)L_n^{1/2}(\rho^2) \ . \tag{15}$$

The integer n gives the number of nodes. These solutions correspond to the small k limit of the Klein-Gordon breathers given earlier by (5).

If λ is nonzero, then defining $S = p_1/\sqrt{|\lambda|}$, equation (13) can be written as

$$S_{,\rho\rho} + \frac{D-1}{\rho}S_{,\rho} + (\omega_2 - \rho^2)S \pm S^3 = 0 \ , \tag{16}$$

where the positive sign is valid for $\lambda > 0$ and the negative for $\lambda < 0$. The solutions are labeled by the single parameter ω_2. If $\lambda > 0$, there are localized nodeless solutions for any $\omega_2 < 3$. In this case high amplitude solutions are more localized than

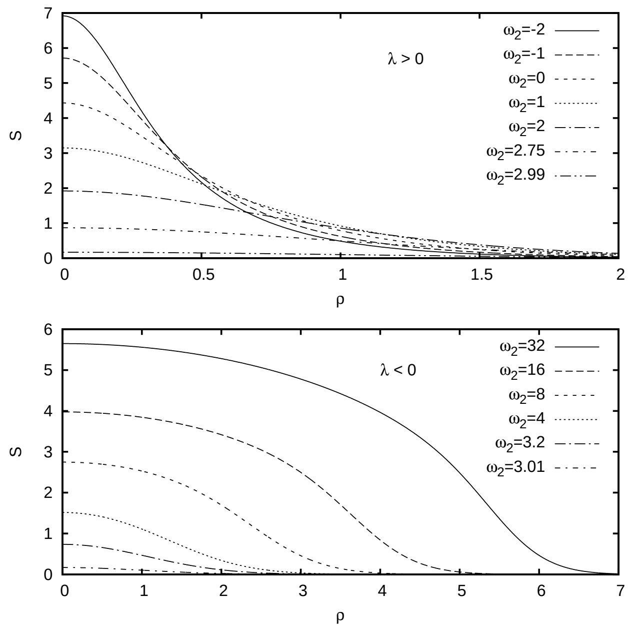

Fig. 4 Radial behavior of solutions of (16) for $\lambda > 0$ attractive, and $\lambda < 0$ repulsive potentials.

small amplitude ones because of the attraction represented by the scalar potential. The radial profiles of a few such solutions are shown on the upper panel of Fig. 4. For $\lambda < 0$ localized nodeless solutions exist for any $\omega_2 > 3$. Then higher amplitude solutions have larger size, and it is natural to call potentials with $\lambda < 0$ "repulsive potentials". Corresponding solutions are shown on the lower panel of Fig. 4. Since by the choice $\kappa = 1$ we have $\varepsilon^2 = k/m$, using (11) and (12) the leading order behavior of the scalar field can be written as

$$\phi = \sqrt{\frac{k|\lambda|}{m}}\, S \cos\left(m\sqrt{1+\frac{k}{m}\omega_2}\; t\right), \tag{17}$$

where S depends on $\rho = r\sqrt{km}$ according to (16).

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