# Some links between general relativity and other parts of physics 

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#### Abstract

Now that General Relativity has become such a central part of modern physics, its geometrical formalism being taught as part of almost all undergraduate physics courses, it is natural to ask: how can its basic concepts and techniques be used to illuminate areas of physics which have no connection with gravity? Another way of asking this question is: are the analogues situations to those occurring in General Relativity? The search for such analogues is of course an old one, but recently, because of advances in technology, these questions have become more topical. In this talk I will illustrate this theme by examples drawn from optics, acoustics, liquid crystals, graphene and the currently popular topic of cloaking.


## 1 Introduction

General Relativity, its mathematical techniques and conceptual framework are by now part of the tool-kit of (almost) all theoretical physicists and at least some pure mathematicians. They have become part of the natural language of physics. Indeed, parts of the subject are passing into mathematics departments. It is natural therefore to ask to what extent can they illuminate other (non-relativistic) areas of physics. It is also the case that the relentless onward progress of technology makes possible analogue experiments illustrating basic ideas in General Relativity. In this talk I will illustrate this ongoing process of unification.

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### 1.1 Dynamical Casimir effect

As a topical example of the relentless progress of technology November last year saw the demonstration in the laboratory [1] some 40 years after the original prediction [2, 3] of a very basic mechanism in semi-classical General Relativity: amplification of vacuum fluctuations in a time-dependent environment. This is the basis of all we believe about inflationary perturbations, Hawking evaporation, Black Hole information "Paradox?" and much of AdS/CFT correspondence etc.

### 1.2 Some previous work

The idea of finding analogue models for general relativistic effects is not new, but the pace has quickened of late. Some important early work was done on cosmic strings modelled by vortices in superfluid helium 4 and by Volovik [4], who noted that the order parameter of some phases of superfluid helium 3 is a triad $\mathbf{e}_{i}$ such that $\mathbf{e}_{i} \cdot \mathbf{e}_{j}=\delta_{i j}$. More recently, the emphasis has shifted to the optics of metamaterials and most recently to graphene. There are also interesting analogies in liquid crystals.

## 2 Shallow water waves

Let's start with a very simple example which will illustrate some basic ideas. If $\eta=$ $\eta(t, x, y)$ is the height of the water above its level when no waves are present and $h=$ $h(x, y)$ the depth of the water, then shallow water waves satisfy the non-dispersive wave equation: (this is the analogue of the Einstein Equivalence Principle)

$$
\left(a_{g} h \eta_{x}\right)_{x}+\left(a_{g} h \eta_{y}\right)_{y}=\eta_{t t},
$$

where $a_{g}$ is the acceleration due to gravity. From now on we adopt units in which $a_{g}=1$. The wave operator coincides with the covariant d'Alembertian

$$
\frac{1}{\sqrt{-g}} \partial_{\mu}\left(\sqrt{-g} g^{\mu v} \partial_{v} \eta\right)=0
$$

with respect to the $2+1$ dimensional spacetime metric

$$
\mathrm{d} s^{2}=-h^{2} \mathrm{~d} t^{2}+h\left(\mathrm{~d} x^{2}+\mathrm{d} y^{2}\right)
$$

Applying ray theory and geometrical optics, one writes

$$
\eta=A e^{-i \omega(t-W(x, y))}
$$

where $A(x, y)$ is slowly varying. To lowest order $W$ satisfies the Hamilton-Jacobi equation

$$
\left(\frac{\partial W}{\partial x}\right)^{2}+\left(\frac{\partial W}{\partial y}\right)^{2}=\frac{1}{h}
$$

and the rays are solutions of

$$
\frac{\mathrm{d} x}{\mathrm{~d} t}=h \frac{\partial W}{\partial x}
$$

Given any static spacetime metric

$$
\mathrm{d} s^{2}=-V^{2} \mathrm{~d} t^{2}+g_{i j} \mathrm{~d} x^{i} \mathrm{~d} x^{j}
$$

the projection $x^{i}=x^{i}(t)$ of light rays, that is characteristic curves of the covariant wave equation or the Maxwell or the Dirac equations, onto the spatial sections are geodesics of the Fermat or optical metric given by

$$
\mathrm{d} s_{o}^{2}=\frac{g_{i j}}{V^{2}} \mathrm{~d} x^{i} \mathrm{~d} x^{j}
$$

In the special case of shallow water waves, the rays are easily seen to be geodesics of the metric

$$
\mathrm{d} s_{o}^{2}=\frac{\mathrm{d} x^{2}+\mathrm{d} y^{2}}{h}
$$

For a linearly shelving beach,

$$
h \propto y \quad y>0
$$

the rays are cycloids, and all ray's strike the shore, i.e. $y=0$, orthogonally. For a quadratically shelving beach,

$$
h \propto y^{2} \quad y>0,
$$

the rays are circles centred on the shore at $y=0$, and again every ray intersects the shore at right angle. In fact the optical metric in this case is

$$
\mathrm{d} s_{o}^{2}=\frac{\mathrm{d} x^{2}+\mathrm{d} y^{2}}{y^{2}}
$$

which is Poincarés metric of constant curvature on the upper half plane. If $x$ is periodically identified, one obtains the the metric induced on a tractrix of revolution in $\mathbb{E}^{3}$, sometimes called the Beltrami trumpet (i.e. $H^{2} / \mathbb{Z}$ ).

For an embedded surface of revolution the induced metric is

$$
\begin{aligned}
h_{i j} \mathrm{~d} x^{i} \mathrm{~d} x^{j} & =\mathrm{d} \rho^{2}+C^{2}(\rho) \mathrm{d} \phi^{2}, & 0 & \leq \phi<2 \pi \\
C^{2}(\rho) & =x^{2}+y^{2}=R^{2}, & \mathrm{~d} \rho^{2} & =\mathrm{d} R^{2}+\mathrm{d} z^{2},
\end{aligned}
$$

with Gauss curvature $K=-\frac{C^{\prime \prime}}{C}$.

For Beltrami's trumpet we have $\rho \geq 0$ and thus

$$
C(\rho)=a \exp \left(-\frac{\rho}{a}\right), \quad K=-\frac{1}{a^{2}}
$$

and if we denote

$$
w=a \phi+i a \exp \left(\frac{\rho}{a}\right)
$$

then the metric is

$$
h_{i j} \mathrm{~d} x^{i} \mathrm{~d} x^{j}=\frac{a^{2}|\mathrm{~d} w|^{2}}{(\mathfrak{I} w)^{2}}
$$

Note that $(\mathrm{d} z / \mathrm{d} \rho)^{2}>0 \Rightarrow$ the embedding can never reach the conformal boundary at $y=0$. This will be significant later.

The optical time for rays to reach the shore in the second example above is infinite. This reminds one of the behaviour of event horizons. In fact there is a rather precise correspondence. The Droste-Schwarzschild metric in isotropic coordinates (setting $G=1=c$ ) is

$$
\mathrm{d} s^{2}=-\frac{\left(1-\frac{m}{2 r}\right)^{2}}{\left(1+\frac{m}{2 r}\right)^{2}} \mathrm{~d} t^{2}+\left(1+\frac{m}{2 r}\right)^{4}\left(\mathrm{~d} x^{2}+\mathrm{d} y^{2}+\mathrm{d} z^{2}\right)
$$

with $r=\sqrt{x^{2}+y^{2}+z^{2}}$. The isotropic radial coordinate $r$ is related to the Schwarzschild radial coordinate $R$ by

$$
R=r\left(1+\frac{m}{2 r}\right)^{2}
$$

The event horizon is at $R=2 m, r=\frac{m}{2}$ If we restrict the Schwarzschild metric to the equatorial plane $z=0$ we obtain

$$
\mathrm{d} s^{2}=-\frac{\left(1-\frac{m}{2 r}\right)^{2}}{\left(1+\frac{m}{2 r}\right)^{2}} \mathrm{~d} t^{2}+\left(1+\frac{m}{2 r}\right)^{4}\left(\mathrm{~d} x^{2}+\mathrm{d} y^{2}\right)
$$

The optical metric is

$$
\mathrm{d} s_{o}^{2}=\frac{\left(1+\frac{m}{2 r}\right)^{6}}{\left(1-\frac{m}{2 r}\right)^{2}}\left(\mathrm{~d} x^{2}+\mathrm{d} y^{2}\right) .
$$

and

$$
h=\frac{\left(r-\frac{m}{2}\right)^{2}}{\left(r+\frac{m}{2}\right)^{6}} r^{4}
$$

We get the analogue of a black hole: a circularly symmetric island whose edge is at $r=\frac{m}{2}$ and away from which the beach shelves initially in a quadratic fashion and ultimately levels out as $r \rightarrow \infty$. Since

$$
\frac{1}{h} \frac{\mathrm{~d} h}{\mathrm{~d} r}=\frac{2}{r-\frac{m}{2}}+\frac{4}{r}-\frac{6}{r+\frac{m}{2}}>0
$$

the beach shelves monotonically.
To obtain a cosmic strings for which the optical metric is a flat cone with deficit angle $\delta=\frac{2 \pi p}{p+1}$ one needs a submerged mountain with

$$
h \propto\left(x^{2}+y^{2}\right)^{\frac{p}{p+1}}
$$

As $p \rightarrow \infty$, we get a parabola of revolution and the optical metric approaches that of an infinitely long cylinder. If $p=1$ the mountain is conical, like a submerged volcano. In physical coordinates $x, y$, the rays are bent, but one may introduce coordinates in which they are flat:

$$
\mathrm{d} s^{2}=\mathrm{d} \tilde{r}^{2}+\tilde{r}^{2} \mathrm{~d} \tilde{\phi}^{2}, \quad 0 \leq \tilde{\phi} \leq \frac{2 \pi}{p+1}
$$

In these coordinates the rays are straight lines. One could multiply these examples to cover such things as cosmic strings, moving water and vortices. To take into account the fact that the Earth is round we replace $\mathbb{E}^{2}$ by $S^{2}$

$$
\mathrm{d} x^{2}+\mathrm{d} y^{2} \rightarrow \mathrm{~d} \theta^{2}+\sin ^{2} \theta \mathrm{~d} \phi^{2}
$$

which gives Einstein's static universe in $2+1$ dimensions. To take into account that it is rotating, we replace the static, i.e. time-reversal invariant metric, by a stationary metric

$$
\mathrm{d} \theta^{2}+\sin ^{2} \theta \mathrm{~d} \phi^{2} \rightarrow \mathrm{~d} \theta^{2}+\sin ^{2} \theta(\mathrm{~d} \phi-\Omega \mathrm{d} t)^{2} .
$$

## 3 Optics and Maxwell's equations

Maxwell's source-free equations in a medium are

$$
\begin{array}{ll}
\operatorname{curl} \mathbf{E}=-\frac{\partial \mathbf{B}}{\partial t}, & \operatorname{div} \mathbf{B}=0 \\
\operatorname{curl} \mathbf{H}=+\frac{\partial \mathbf{D}}{\partial t}, & \operatorname{div} \mathbf{D}=0
\end{array}
$$

or if $\left(\varepsilon_{i j k}= \pm 1,0\right)$

$$
\begin{aligned}
& F=-E_{i} \mathrm{~d} t \wedge \mathrm{~d} x^{i}+\frac{1}{2} \varepsilon_{i j k} B_{i} \mathrm{~d} x^{j} \wedge \mathrm{~d} x^{k} \\
& G=\quad H_{i} \mathrm{~d} t \wedge \mathrm{~d} x^{i}+\frac{1}{2} \varepsilon_{i j k} D_{i} \mathrm{~d} x^{j} \wedge \mathrm{~d} x^{k}
\end{aligned}
$$

$$
\mathrm{d} F=0=\mathrm{d} G
$$

In what follows it will be important to realise that these equations hold in any coordinate system and they do not require the introduction of a spacetime metric.

However to "close the system", one must relate $F$ to $G$ by means of a "constitutive equation". If the medium is assumed to be static and linear, then

$$
D_{i}=\varepsilon_{i j} E_{j}, \quad B_{i}=\mu_{i j} H_{j}
$$

where $\varepsilon_{i j}$ is the dielectric permittivity tensor and $\mu_{i j}$ the magnetic permeability tensor. If they are assumed symmetric: $\varepsilon_{i j}=\varepsilon_{j i}, \mu_{i j}=\mu_{j i}$, then $\mathscr{E}=\frac{1}{2}\left(E_{i} D_{i}+H_{i} B_{i}\right)$ may be regarded as the energy density and $\mathbf{S}=\mathbf{E} \times \mathbf{H}$ the energy current or Poynting vector since Maxwell's equations imply

$$
\operatorname{div} \mathbf{S}+\frac{\partial \mathscr{E}}{\partial t}=0
$$

"In olden days a glimpse of stocking was thought of as something shocking" and certainly $\mu_{i j}$ and $\varepsilon_{i j}$ were assume positive definite "but now", with the advent of nanotechnology and the construction of metamaterials "anything goes".

### 3.1 Left-handed light

As long ago as 1964, V.G. Vestilago pointed out that isotropic substances with with $\mu_{i j}=\mu \delta_{i j}, \varepsilon_{i j}=\varepsilon \delta_{i j}$ and for which

$$
\mu<0, \quad \varepsilon<0
$$

give rise to left-handed light moving in a medium with a negative refractive index. In 2001, R.A. Shelby, D.R. Smith and S. Schutz [5] produced this effect for microwave frequencies. In 2002, D.R. Smith, D. Schurig and J.B. Pendry [6] appeared to have produced this effect in the laboratory.

Assuming a spacetime dependence proportional to an arbitrary function of $\mathbf{k} \cdot \mathbf{x}-$ $\omega t$, with $\omega>0$, one finds

$$
\begin{array}{ll}
\mathbf{k} \times \mathbf{E}=\omega \mathbf{B}, & \mathbf{k} \times \mathbf{H}=-\omega \mathbf{D} \\
\mathbf{k} \times \mathbf{E}=\mu \omega \mathbf{H}, & \mathbf{k} \times \mathbf{H}=-\varepsilon \omega \mathbf{E}
\end{array}
$$

It is always the case that $(\mathbf{E}, \mathbf{H}, \mathbf{S})$ form a right-handed orthogonal triad but if both $\mu$ and $\varepsilon$ are negative then $(\mathbf{E}, \mathbf{H}, \mathbf{k})$ form a left-handed orthogonal triad and so $\mathbf{S}$ and $\mathbf{k}$ are anti-parallel rather than parallel as is usually the case. Since the wave vector $\mathbf{k}$ must be continuous across a junction between a conventional medium and an exotic medium with $\mu<0, \varepsilon<0$, this gives rise to backward bending light.

The speed of propagation $v=\frac{1}{n}$, where $n$ is the refractive index, is given by

$$
v^{2}=\frac{\omega^{2}}{\mathbf{k}^{2}}=\frac{1}{\mu \varepsilon}
$$

it is natural to take the negative square root to get the refractive index

$$
n=-\frac{1}{\sqrt{\mu \varepsilon}}
$$

Given a spacetime metric $g_{\mu \nu}$ one has a natural way of specifying a constitutive relation:

$$
G=\star_{g} F,
$$

where $\star_{g}$ denotes the Hodge dual with respect to the spacetime metric $g$ such that $\star_{g} \star_{g}=-1$. If

$$
\mathrm{d} s^{2}=-V^{2}\left(x^{k}\right) \mathrm{d} t^{2}+g_{i j}\left(x^{k}\right) \mathrm{d} x^{i} \mathrm{~d} x^{j}
$$

Tamm [7], Skrotskii, [8] and Plebanski [9] showed that

$$
\mu_{i j}=\varepsilon_{i j}=\sqrt{\frac{\operatorname{det} g_{l m}}{V^{2}}} g^{i j}
$$

A medium with $\mu_{i j}=\varepsilon_{i j}$ is said to be impedance matched. A similar result holds for resistivity problems such as that Calderon [10] encountered oil prospecting

$$
\begin{gathered}
\nabla \cdot \mathbf{j}=0, \quad \mathbf{E}=-\boldsymbol{\nabla} \phi, \quad j_{i}=\sigma_{i j} E_{j}, \\
\partial_{i}\left(\sigma_{i j} \partial_{j} \phi\right)=0 \quad \Rightarrow \quad \nabla_{g}^{2} \phi=\frac{1}{\sqrt{g}} \phi_{i}\left(\sqrt{g} g^{i j} \partial_{j} \phi\right)=0,
\end{gathered}
$$

with

$$
\sigma_{i j}=\sqrt{g} g^{i j}, \quad g_{i j}=\left(\operatorname{det} \sigma_{i j}\right) \rho_{i j}
$$

If

$$
\sigma_{i j}=\frac{1}{z} \delta_{i j}
$$

we get Poincaré metric on upper half space model of hyperbolic or Lobachevsky space $H^{2}$

$$
\mathrm{d} s^{2}=\frac{\mathrm{d} x^{2}+\mathrm{d} y^{2}+d z^{2}}{z^{2}}
$$

The conformal boundary is a perfect conductor.
In physics we may choose either the West Coast signature convention $(-,+,+,+)$, so that $g_{t t}<0$ and $g_{i j}$ is positive definite or the East Coast convention $(+,-,-,-)$ for which $g_{t t}>0$ and $g_{i j}$ is negative definite. By Sylvester's law of inertia the signature is locally constant, however running between the East Coast and the West coast there must be a curve on which the spacetime signature flips (as originally suggested in a different context by Arthur Eddington in 1922). Clearly, light passing from Coast to Coast will get bent back.

By Fermat's principle electromagnetic waves move along geodesics of the optical metric

$$
\mathrm{d} s_{o}^{2}=V^{-2} g_{i j} \mathrm{~d} x^{i} \mathrm{~d} x^{j}
$$

but this is invariant under signature change.

### 3.2 Zermelo-Randers-Finsler geometry

If time reversal symmetry is broken a stationary metric may be cast in three different forms [11]:

$$
\begin{aligned}
\mathrm{d} s^{2} & =-U\left(\mathrm{~d} t+\omega_{i} \mathrm{~d} x^{i}\right)^{2}+\gamma_{i j} \mathrm{~d} x^{i} \mathrm{~d} x^{j} \\
& =U\left[-\left(\mathrm{d} t-b_{i} \mathrm{~d} x^{i}\right)^{2}+a_{i j} \mathrm{~d} x^{i} \mathrm{~d} x^{j}\right] \\
& =\frac{U}{1-h_{i j} W^{i} W^{j}}\left[-\mathrm{d} t^{2}+h_{i j}\left(\mathrm{~d} x^{i}-W^{i} \mathrm{~d} t\right)\left(\mathrm{d} x^{j}-W^{j} \mathrm{~d} t\right)\right]
\end{aligned}
$$

Fermat's principle for light rays now generalises to Zermelo's problem: minimize the travel time of a boat moving with fixed speed with respect to a Riemannian metric $h_{i j}$ in the presence of a "wind" $W^{i}$.

One may also think of the problem as one of a particular type of Finsler geometry considered first by Randers with a Finsler function of homogeneous degree one in velocity $v^{i}=\frac{\mathrm{d} x^{i}}{\mathrm{~d} \lambda}$ defining a line element $\mathrm{d} s=F \mathrm{~d} \lambda$, given by

$$
F=\sqrt{a_{i j} v^{i} v^{j}}+b_{i} v^{i}
$$

Alternatively, one may think of a charged particle of unit mass and unit charge, moving on a Riemannian manifold with metric $a_{i j}$ and magnetic field $B_{i j}=\partial_{i} b_{j}-$ $\partial_{i} b_{j}$. In General Relativity, this is gravito-magnetism verified recently by the GP-B satellite experiment.

In the absence of time reversal symmetry there is a magneto-electric effect first predicted by L. Landau and E.M. Lifshitz in 1956 and exhibited for instance by $\mathrm{Cr}_{2} \mathrm{O}_{3}$ :

$$
\begin{gathered}
B_{i}=\mu_{i j} H_{j}+\alpha_{j i} E_{j}, \quad D_{i}=\varepsilon_{i j} E_{j}+\alpha_{i j} H_{j} \\
\mathscr{E}=\frac{1}{2} \mu_{i j} H_{i} H_{j}+\alpha E_{i} H_{j}+\frac{1}{2} \varepsilon_{i j} E_{i} E_{j}
\end{gathered}
$$

If we take as constitutive relation $G=\star_{g} F$, then $\mu_{i j}, \varepsilon_{i j}$ and $\alpha_{i j}$ may be read off from the spacetime metric.

In a moving medium, a typical sound or light wave satisfies

$$
\left[\left(\partial_{t}-W^{i} \partial_{i}\right)^{2}-h^{i j} \partial_{i} \partial_{j}\right] u=0
$$

The rays solve the Zermelo problem with wind $W^{i}$. For sound waves this is known to explain the curious (and irritating) propagation of traffic noise. The rays behave like charged magnetic particles, the magnetic field being given by the vertical gradient of the horizontal wind. Of course a vertical gradient in temperature and hence refractive index will also provide an anti-mirage effect. This produces a curved metric $h_{i j}$. Claude Warnick and I have recently modelled this by a charged particle moving in a magnetic field on the upper half plane [12].

For black or white holes Zermelo picture is equivalent to the use of PainlevéGullstrand coordinates. Here is a low-tech example involving just a kitchen sink [13]. The ripples are surface tension ripples.


Fig. 1 Here is the Mach cone.

### 3.3 Invisibility cloaks

Designing invisibility cloaks, analogue black holes, etc. using metamaterials and transformation optics. The basic idea is to start with a metric and read off $\varepsilon_{i j}$ and $\mu_{i j}$. The metric could even be flat and obtained by a local diffeomorphism from the flat metric by which a beam or pencil of parallel straight lines in Cartesian coordinates are taken to the desired set of light rays in an impedance matched metamaterial medium. This technique has been much exploited by Pendry, Leonhardt and their collaborators and followers recently.

As pointed out by Uhlmann and others, similar problems arise in Calderon's inverse problem: given a measurement of $\mathbf{E}$ and $\phi$ on the boundary of some domain, can you determine uniquely the conductivity in the interior or can a reservoir of oil be invisible to the prospector?

In general one needs anisotropic materials.
To obtain an isotropic metamaterial medium the local diffeomorphisms should be conformal. The oldest and best known example of this is Maxwell's fish eye lens
which makes use use of Hipparchus's stereographic projection. This is the basis of the Luneburg lens [14].

A variant due to Minano [15] pulls back the round metric on $S^{2}(\theta, \phi)$ to $R^{2}(x, y)$ using

$$
x=\left(\frac{1-\sin \theta}{\cos \theta}\right)^{\frac{1}{p}} \cos \left(\frac{\phi}{p}\right), \quad y=\left(\frac{1-\sin \theta}{\cos \theta}\right)^{\frac{1}{p}} \sin \left(\frac{\phi}{p}\right)
$$

to get

$$
\mathrm{d} s_{o}^{2}=\mathrm{d} \theta^{2}+\cos ^{2} \theta \mathrm{~d} \phi^{2}=n^{2}\left(\mathrm{~d} x^{2}+\mathrm{d} y^{2}\right), \quad n=2 p^{2} \frac{r^{p-1}}{r^{2 p}+1} .
$$

To get a black hole start again with the Droste-Schwarzschild metric in isotropic coordinates

$$
\begin{aligned}
\mathrm{d} s^{2} & =-\frac{\left(1-\frac{m}{2 r}\right)^{2}}{\left(1+\frac{m}{2 r}\right)^{2}} \mathrm{~d} t^{2}+\left(1+\frac{m}{2 r}\right)^{4}\left(\mathrm{~d} x^{2}+\mathrm{d} y^{2}+\mathrm{d} z^{2}\right), \\
n=\mu=\varepsilon & =\left(1+\frac{m}{2 r}\right)^{3}\left(1-\frac{m}{2 r}\right)^{-1} .
\end{aligned}
$$

The original cloak construction by Uhlmann works like this. We consider a spherical shell or solid annulus $a<r<2 a$ in $r, \theta, \phi$ space and map it onto the punctured disc $0<\tilde{r}<2 a$ by

$$
\tilde{r}=2(r-a), \quad \tilde{\theta}=\theta, \quad \tilde{\phi}=\phi .
$$

The map is the identity: $r=\tilde{r}$ for $r>2 a, \tilde{r}>2 a$. Now pull back the flat metric $\mathrm{d} \tilde{r}^{2}+\tilde{r}^{2}\left(\mathrm{~d} \tilde{\theta}^{2}+\sin ^{2} \tilde{\theta} \mathrm{~d} \tilde{\phi}^{2}\right)$ and straight lines in $\tilde{r}, \tilde{\theta}, \tilde{\phi}$ space:

$$
\begin{gathered}
\mathrm{d} s^{2}=4 \mathrm{~d} r^{2}+4(r-a)^{2}\left(\mathrm{~d} \theta^{2}+\sin ^{2} \theta \mathrm{~d} \phi^{2}\right) \\
\varepsilon=\mu=\operatorname{diag}\left(2(r-a)^{2} \sin \theta, 2 \sin \theta, \frac{2}{\sin \theta}\right)
\end{gathered}
$$

No light ray (or electric current) enters the solid ball $r<a$.
The construction just given is strikingly similar to that used in the hole argument which played a big part in Einstein's understanding of the concept of general covariance and his search for covariant field equations in the years from 1913 to 1915 [16].

It is remarkable that what hitherto has been of interest almost exclusively to philosophers and historians of science is now at the centre of a new technology!

### 3.4 Hyperbolic metamaterials and two-time physics

Another possibly are hyperbolic metamaterials for which $\varepsilon_{i j}$ is an indefinite matrix. The dispersion relation for a bi-refringent medium with $\mu_{i j}=\delta_{i j}$ is a quartic cone of two sheets:

$$
\left(\frac{k_{x}^{2}}{n_{o}^{2}}+\frac{k_{y}^{2}}{n_{o}^{2}}+\frac{k_{z}^{2}}{n_{o}^{2}}-\frac{\omega^{2}}{c^{2}}\right)\left(\frac{k_{x}^{2}}{n_{e}^{2}}+\frac{k_{y}^{2}}{n_{e}^{2}}+\frac{k_{z}^{2}}{n_{o}^{2}}-\frac{\omega^{2}}{c^{2}}\right)=0
$$

with $n_{o}^{2}=\varepsilon_{z}, n_{e}^{2}=\varepsilon_{x}=\varepsilon_{y}$. Exceptional electromagnetic waves in a uniaxial medium thus obey

$$
\frac{1}{c^{2}} \frac{\partial^{2} E}{\partial t^{2}}=\frac{1}{\varepsilon_{1}} \frac{\partial^{2} E}{\partial z^{2}}+\frac{1}{\varepsilon_{2}}\left(\frac{\partial^{2} E}{\partial x^{2}}+\frac{\partial^{2} E}{\partial y^{2}}\right)
$$

The idea is [17] that dipole-moments in some crystals such as $\alpha$ quartz interact with lattice vibrations to form phonon-polariton modes called restrahlen bands in the mid infra-red region for which both $\varepsilon_{1}$ and $\varepsilon_{2}$ can become negative. Moreover because of crystal anisotropy $\varepsilon_{1}$ and $\varepsilon_{2}$ change sign at slightly different temperatures. This would allow an effective two-time physics.

In a model in a layered composite dielectric material

$$
\varepsilon_{2}=n_{m}+\left(1-n_{m}\right) \varepsilon_{d}, \quad \varepsilon_{1}=\frac{\varepsilon_{m} \varepsilon_{d}}{\left(1-n_{m}\right) \varepsilon_{m}+n_{m} \varepsilon_{d}}
$$

where the subscripts $d$ and $m$ stand for dielectric and metal respectively and $\varepsilon_{m}$ is frequency dependent and can be negative. $n_{m}$ is the volume fraction of metal. In a simple Drude model

$$
\varepsilon_{m}=1-\frac{\omega_{p}^{2}}{\omega^{2}+i \omega \gamma}
$$

with $\frac{\gamma}{\omega_{p}}$ being small. If $n_{m} \ll 1$ we have

$$
\varepsilon_{2} \approx \varepsilon_{d}-\frac{n_{m} \omega_{p}^{2}}{\omega^{2}+i \omega \gamma}, \quad \varepsilon_{1} \approx \varepsilon_{d}
$$

## 4 Chiral nematics

Rather than consider artificial impedance matched or hyperbolic metamaterials, we may consider realistic substances such as chiral nematics in their helical phase. Up to a divergence the Frank-Oseen free energy is

$$
F=\frac{1}{2} \int\left(\left|\nabla^{q} \mathbf{n}\right|^{2}-\lambda(\mathbf{n} \cdot \mathbf{n}-1)\right) \mathrm{d}^{3} x
$$

$$
\nabla_{i}^{q} n_{j}=\partial_{j} n_{j}+q \varepsilon_{i j k} n_{k}
$$

where $\nabla^{q}$ is a Euclidean metric preserving connection with torsion. The free energy density would vanish if $\mathbf{n}$ were covariantly constant with respect to $\nabla^{q}$, i.e. $\nabla_{i}^{q} n_{j}=0$. But rather like an anti-ferromagnet it is frustrated since

$$
\left(\nabla_{i}^{q} \nabla_{j}^{q}-\nabla_{j}^{q} \nabla_{i}^{q}\right) n_{k} \neq 0
$$

The substance may adopt a compromise configuration called the helical phase which satisfies the second order equations but not the first order Bogomolnyi type equation

$$
\mathbf{n}=(\cos (p z), \sin (p z), 0)
$$

Optics in a nematic liquid crystal is governed by Fermat's principle using the JoetsRibotta metric

$$
\mathrm{d} s_{o}^{2}=n_{e}^{2} \mathrm{~d} \mathbf{x}^{2}+\left(n_{o}^{2}-n_{e}^{2}\right)(\mathbf{n} \cdot \mathrm{d} \mathbf{x})^{2},
$$

where $n_{o}$ is the refractive index of the ordinary ray and $n_{e}$ that of the extra-ordinary ray.

Introducing 3 one-forms with Maurer-Cartan relations

$$
\begin{array}{ll}
\lambda^{1}=\cos (p z) \mathrm{d} x+\sin (p z) \mathrm{d} y, & \mathrm{~d} \lambda^{1}=\lambda^{3} \wedge \lambda^{2}, \\
\lambda^{2}=\cos (p z) \mathrm{d} x-\sin (p z) \mathrm{d} y, & \mathrm{~d} \lambda^{2}=\lambda^{3} \wedge \lambda^{1}, \\
\lambda^{3}=p \mathrm{~d} z, & \mathrm{~d} \lambda^{3}=0,
\end{array}
$$

we find the Joets-Ribotta metric to be

$$
\mathrm{d} s_{o}^{2}=n_{o}^{2}\left(\lambda^{1}\right)^{2}+n_{e}^{2}\left(\lambda^{2}\right)^{2}+\frac{n_{e}^{2}}{p^{2}}\left(\lambda^{3}\right)^{2} .
$$

This is a left-invariant metric on $\tilde{E}(2)$, the universal cover of the two-dimensional Euclidean group $E(2)$ whose Lie algebra $e(2)$ is of Type $V I I_{0}$ in Bianchi's classification.

Thus the helical phase of chiral nematic crystals gives rise to a static Bianchi $V I I_{0}$ cosmology:

$$
\mathrm{d} s^{2}=-\mathrm{d} t^{2}+n_{o}^{2}\left(\lambda^{1}\right)^{2}+n_{e}^{2}\left(\lambda^{2}\right)^{2}+\frac{n_{e}^{2}}{p^{2}}\left(\lambda^{3}\right)^{2}
$$

and one may, and we did, use all the standard tools of general relativistic cosmology to describe its optical and electromagnetic properties, including solving Maxwell's equations, applying the Floquet-Bloch theorem and the associated Mathieu-Hill equation.

## 5 Gravitational kinks

The topology of a Lorentzian metric may be (partially) captured by a direction field $n^{i}$. Given a Riemannian metric $g_{i j}^{R}$, and a unit direction field $n^{i}$ such that $g_{i j}^{R} n^{i} n^{j}=1$ we may construct a Lorentzian metric $g_{i j}^{L}$ via

$$
g_{i j}^{L}=g_{i j}^{R}-\frac{1}{\sin ^{2} \alpha} n_{i} n_{j}, \quad g_{L}^{i j}=g_{R}^{i j}-\frac{1}{\cos ^{2} \alpha} n^{i} n^{i}, \quad n_{i}=g_{i j}^{R} n^{j}
$$

Conversely, given $g_{i j}^{L}$ and $g_{i j}^{R}$ we may reconstruct $n_{i}$ up to a sign. Fixing the sign amounts to fixing a time orientation. In what follows we will choose $g_{i j}^{R}$ to be the usual flat Euclidean metric:

$$
\mathrm{d} s_{L}^{2}=g_{L}^{i j} \mathrm{~d} x^{i} \mathrm{~d} x^{j}=\mathrm{d} \mathbf{x}^{2}-\frac{1}{\cos ^{2} \alpha}(\mathbf{n} \cdot \mathrm{~d} \mathbf{x})^{2}
$$

Given a closed surface enclosing a domain $D$, Finkelstein and Misner quantified the notion of tumbling light cones - the light cone tips over on $\Sigma=\partial D-$ by introducing a kink number which counts how many times the light cone tips over on $\Sigma$. The outward unit normal $\boldsymbol{v}$ gives a 2 -dimensional cross section of the four-dimensional bundle $S(\Sigma)$ of unit 3-vectors over $\Sigma$. In the orientable case, the director field gives another 2-dimensional cross section of $S(\Sigma)$. The kink number, $\operatorname{kink}\left(\Sigma, g^{L}\right)$, is number of intersections of these two sections with attention paid to signs. In the non-orientable case, one considers the bundle of directions. If the Lorentzian metric is non-singular, we have

$$
\chi(D)=\operatorname{kink}\left(\partial D, g^{L}\right)
$$

For planar domains $\operatorname{kink}\left(\partial D, g^{L}\right)$ is the obvious winding number.
Disclination line is defined by

$$
\mathbf{n}=(\cos (s \phi), \sin (s \phi), 0), \quad \phi=\arctan \left(\frac{y}{x}\right)
$$

where $s \in \mathbb{Z} \cup \mathbb{Z}+\frac{1}{2}$. If $s$ is half integral, then we just have a direction field, not a vector field.

$$
\mathbf{n} \cdot \mathrm{d} \mathbf{x}=\cos ((s-1) \phi) \mathrm{d} r+\sin ((s-1) \phi) r \mathrm{~d} \phi .
$$

For $\alpha=\frac{\pi}{2}$ we get

$$
\left.\mathrm{d} s_{L}^{2}=g_{i j}^{L} \mathrm{~d} x^{i} \mathrm{~d} x^{j}=-\cos (2(s-1) \phi)\left(\mathrm{d} r^{2}-r^{2} \mathrm{~d} \phi^{2}\right)\right)-2 \sin (2(s-1) \phi) r \mathrm{~d} r \mathrm{~d} \phi .
$$

Moving around a circle $r=$ constant, the radial coordinate is timelike and the angular coordinate spacelike or vice versa depending upon the sign of $\cos (2(s-1) \phi)$ (tumbling light cones). The metric components $g_{i j}^{L}$ are finite and $\operatorname{det} g_{i j}^{L}=-r^{2} \Rightarrow$ metric non-singular if $r>0$.

### 5.1 Bloch walls

If parity symmetry holds then a typical free energy functional takes the form

$$
F[\mathbf{M}]=\frac{1}{2} \int \mathrm{~d} x\left(\alpha_{i j} \partial_{i} \mathbf{M} \cdot \partial_{j} \mathbf{M}+\beta_{i j} M_{i} M_{j}\right)
$$

In the uniaxial case with the easy direction along the third direction $\alpha_{i j}=\operatorname{diag}\left(\alpha_{1}, \alpha_{1}, \alpha_{2}\right)$, $\beta_{i j}=\operatorname{diag}(\beta, \beta, 0)$. For a domain wall separating a region $x \ll-1$ and with $\mathbf{M}$ pointing along the positive $3^{\text {rd }}$ direction, from the region $x \gg+1$ where it points along the negative $3^{\text {rd }}$ direction, we have

$$
\mathbf{M}=M(0, \sin \theta(x), \cos \theta(x)), \quad M=\text { constant }
$$

and finds that $\theta$ must satisfy the quadrantal pendulum equation, $l=\sqrt{\frac{\alpha_{1}}{\beta}}$,

$$
\theta^{2}-\frac{1}{l^{2}} \sin ^{2} \theta=\text { constant }{ }^{\prime}
$$

If we impose the boundary condition that $\theta \rightarrow 0$ as $x \rightarrow-\infty$ and $\theta \rightarrow \pi$ as $x \rightarrow$ $+\infty$, then constant ${ }^{\prime}=0$ and

$$
\cos \theta=-\tanh \left(\frac{x}{l}\right)
$$

The Lorentzian metric (if $\alpha=\frac{\pi}{2}$ ) is

$$
\mathrm{d} s^{2}=g_{i j}^{L} \mathrm{~d} x^{i} \mathrm{~d} x^{j}=\mathrm{d} x^{2}+\cos (2 \theta)\left(\mathrm{d} y^{2}-\mathrm{d} z^{2}\right)-2 \sin 2 \theta \mathrm{~d} z \mathrm{~d} y
$$

This closely resembles our previous examples and clearly exhibits the phenomenon of tumbling light cones. We note, en passant, that in principle the tensor $\alpha_{i j}$ could itself vary with position. If so, we might interpret it in terms of an effective metric $g_{i j}$ with inverse $g^{i j}$ and $g=\operatorname{det} g_{i j}$ obeying

$$
\alpha_{i j}=\sqrt{g} g^{i j}
$$

### 5.2 Liquid crystal droplets

The normal $v_{i}=\partial_{i} S$ to the surface $S=$ constant of a droplet of anisotropic nematic phase inside a domain $D$ with unit outward normal $\boldsymbol{v}$ surrounded by an isotropic phase satisfies the constant angle condition

$$
\mathbf{n} \cdot \boldsymbol{v}=\cos \alpha=\text { constant }
$$

That is

$$
\boldsymbol{v} \cdot \boldsymbol{v}-\frac{1}{\cos ^{2} \alpha}(\boldsymbol{v} \cdot \mathbf{n})(\boldsymbol{v} \cdot \mathbf{n})=0=g_{L}^{i j} v_{i} v_{j}=g_{L}^{i j} \partial_{i} S \partial_{j} S
$$

The surface $\partial D$ of the droplet $\partial D$ is a null-hypersurface or wave surface (a solution of the zero-rest-mass Hamilton-Jacobi equation)

Taking the $z$-coordinate as time so time runs vertically upwards and making the ansatz

$$
S=\frac{z}{\sin \alpha}+W(x, y), \quad \nabla W \cdot \nabla W=1
$$

Simple solutions of this Eikonal equation are given by sandpiles with $\frac{\pi}{2}-\alpha$ the angle of repose.

These describe Bitter domains in a ferromagnetic film with $\mathbf{n}=\frac{\mathbf{M}}{|\mathbf{M}|}$ with normal $\boldsymbol{v}$ and boundary condition $\mathbf{M} \cdot \boldsymbol{v}=0$.

$$
\begin{gathered}
\boldsymbol{\nabla} \cdot \mathbf{M}=0, \quad|\mathbf{M}|=\text { constant. } \\
\boldsymbol{\nabla} \cdot \mathbf{n}=0 \quad \Rightarrow \quad n_{x}=\partial_{y} \psi, \quad n_{y}=-\partial_{x} \psi \quad|\nabla \psi|=1 .
\end{gathered}
$$

The axisymmetric solution is the spiral wave surface swept out by the involute of a circle, a helical developable:

$$
S= \pm \frac{z}{\sin \alpha} \pm a\left(\sqrt{\frac{r^{2}}{a^{2}}-1}-\arctan \left(\sqrt{\frac{r^{2}}{a^{2}}-1}\right)\right) \pm a \phi
$$

### 5.3 Helical phase

We make the ansatz

$$
S=F(z)+x \cos \theta+y \sin \theta,
$$

where $F(z)$ solves the quadrantal pendulum equation

$$
\cos ^{2}(\theta-p z)-\cos ^{2} \alpha=\left(\cos \alpha \frac{\mathrm{d} F}{\mathrm{~d} z}\right)^{2}
$$

Thus

$$
F=\frac{1}{\cos \alpha} \int \mathrm{~d} z \sqrt{\cos ^{2}(\theta-p z)-\cos ^{2} \alpha}
$$

The surface is ruled by horizontal straight lines making a constant angle $\theta$ with the $x$-axis and is bounded by $|p z-(\theta+n \pi)|<\alpha, n \in \mathbb{Z}$. In other words it is horizontal cylinder or tube. The angle that the director $\mathbf{n}$ makes with the fixed direction $(\cos \theta, \sin \theta, 0)$ cannot be less than $\alpha$.

## 6 Graphene

The hexagonal Graphene "lattice" in $\mathbf{x}$-space has a hexagonal Brillouin zone in the dual $\mathbf{p}$-space and is the sum of two triangular (true) lattices, A and B in $\mathbf{x}$-space. Each lattice has a Fermi surface in $\mathbf{p}$-space and these two Fermi surfaces, governing the conduction and valence bands, touch in two conical Dirac points inside a Brillouin zone. Thus the dispersion relation for small $\mathbf{p}$ is

$$
E= \pm|\mathbf{p}|
$$

Low energy excitations are governed by

$$
E \Psi=\boldsymbol{\sigma} \cdot \mathbf{p} \Psi
$$

where the two-component $\Psi$ has two pseudo-spin states.
But this is the massless Dirac equation! (cf. [18])
On a curved graphene sheet it becomes the Dirac equation on a curved surface $\Sigma \subset \mathbb{E}^{3}$ in Euclidean 3-space with metric

$$
\mathrm{d} s^{2}=-\mathrm{d} t^{2}+h_{i j} \mathrm{~d} x^{i} \mathrm{~d} x^{j}, \quad i, j=1,2
$$

where $h_{i j}$ is the induced metric.
Since the massless Dirac equation is conformally invariant we may think of this metric on $R \times \Sigma$ as the optical metric of a static metric with $g_{t t} \neq$ constant.

If $\Sigma$ is a Beltrami trumpet with metric of constant negative curvature, we have the optical metric of identified Rindler spacetime. This is also near horizon optical geometry of a general 2-dimensional black hole. Unfortunately we cannot find an isometric embedding of $H^{2} / \mathbb{Z}$ into $\mathbf{E}^{3}$ all the way down to $y=0$, the horizon.

This is a general problem: a global theorem of Hilbert forbids isometric embeddings of complete surface of constant negative curvature into Euclidean space $\mathbb{E}^{3}$.

More generally we may consider a BTZ black hole [19]:

$$
\begin{aligned}
\mathrm{d} s_{B T Z}^{2} & =-\Delta \mathrm{d} t^{2}+\frac{d r^{2}}{\Delta}+r^{2}\left(\mathrm{~d} \phi-\frac{J}{2 r^{2}} \mathrm{~d} t\right)^{2} \\
\Delta(r) & =\frac{r^{2}}{l^{2}}-M+\frac{J^{2}}{4 r^{2}}
\end{aligned}
$$

Zermelo metric:

$$
h_{i j} \mathrm{~d} x^{i} \mathrm{~d} x^{j}=\frac{\mathrm{d} r^{2}}{\Delta^{2}}+\frac{r^{2}}{\Delta} \mathrm{~d} \phi^{2}
$$

Wind

$$
W^{i} \partial_{i}=\frac{J}{2 r^{2}} \partial_{\phi}
$$

In the near horizon limit $h_{i j}$ is of Beltrami trumpet form.
The massless Dirac equation in the Zermelo frame is

Some links between general relativity and other parts of physics

$$
\begin{aligned}
{\left[\gamma^{1}\left(\partial_{\rho}+\frac{1}{2} \frac{C^{\prime}}{C}\right)+\gamma^{2} \frac{1}{C} \partial_{\phi}+\gamma^{0}\left(\partial_{t}+W \partial_{\phi}\right)+\frac{1}{4} \gamma^{0} \gamma^{1} \gamma^{2} C W^{\prime}\right] \Psi } & =0 \\
\gamma^{0}=i \sigma_{2}, \quad \gamma^{1}=\sigma_{1}, \quad \gamma^{2}=\sigma_{3} . \quad \gamma^{0} \gamma^{1} \gamma^{2} & =1
\end{aligned}
$$

and we get a position dependent "mass-like" term and a connection term. If $\Psi \propto$ $e^{-i \omega t+i m \phi}$ we have that

$$
-i e A_{0}=i m W, \quad \Rightarrow \quad e A_{0}=-m W
$$

A stationary Zermelo metric induces in the Dirac equation an effective, position dependent radial electric field.

We could have done this calculation in the Randers frame. The detailed form of the Randers metric is considerably more complicated. The embedding is qualitatively similar, but different.

More interestingly, because now the roles of $t$ and $\phi$ have essentially been interchanged, we now find that there is an effective magnetic vector potential in the Dirac equation. Therefore, the magnetic vector potential in the Randers frame appears as an electric potential in the Zermelo one.

Since the Zermelo and Randers frames are in relative motion, this is just a manifestation of the fact that under boosts magnetic and electric fields transform into themselves. In either case, these effects could be mimicked by applying external electric (Zermelo) or magnetic (Randers) fields to the two different graphene sheets.

### 6.1 Cold Atoms

Instead of graphene one may consider, and people have discussed, metrics in the context of cold atoms in Bose-Einstein condensates [20].

## 7 Conclusion and Propects

In this talk I have described on some areas of non-gravitational physics where analogues of basic ideas in general relativity come into play. They include

1. Dynamic Casimir effect
2. Water and sound waves
3. Cloaking and other devices using metamaterials
4. Nematic liquid crystals
5. Graphene

Other areas not covered include

1. Bose-Einstein condensate
2. Dirac metals
3. Smectcs and blue phases in liquid crystals

## References

1. C.M. Wilson, G. Johansson, A. Pourkabirian, et al., Observation of the dynamical Casimir effect in a superconducting circuit, Nature 479, 376 (2011)
2. G.T. Moore, Quantum theory of the electromagnetic field in a variable-length one-dimensional cavity, J. Math. Phys. 11, 2679 (1970)
3. S.A. Fulling, P.C.W. Davies, Radiation from a moving mirror in two dimensional space-time Conformal anomaly, Proc. R. Soc. London, Ser. A 348, 393 (1976)
4. G.E. Volovik, The Universe in a Helium Droplet, International Series of Monographs on Physics, vol. 117 (Oxford University Press, Oxford; New York, 2003)
5. R.A. Shelby, D.R. Smith, S. Schutz, Experimental verification of a negative index of refraction, Science 292, 77 (2001)
6. D.R. Smith, D. Schurig, J.B. Pendry, Negative refraction of modulated electromagnetic waves, Appl. Phys. Lett. 81, 2713 (2002)
7. I.E. Tamm, Elektrodinamika anizotropnoj sredy v special'noj teorii otnositel'nosti, Zh. Rus. Fiz.-Khim. Obshchestva, Otd. Fiz 56, 248 (1924)
8. G.V. Skrotskii, On the influence of gravity on the light propagation, Dokl. Akad. Nauk. SSSR 114, 73 (1957). Soviet Physics Doklady 2, 226 (1957)
9. J. Plebański, Electromagnetic waves in gravitational fields, Phys. Rev. 118, 1396 (1960)
10. A.P. Calderón, On an Inverse Boundary Value Problem, in Seminar in Numerical Analysis and its Applications to Continuum Physics, ed. by W.H. Meyer, M.A. Raupp (Sociedade Brasileira de Matemática, Rio de Janeiro, 1980), pp. 65-73
11. G.W. Gibbons, C.A.R. Herdeiro, C.M. Warnick, M.C. Werner, Stationary metrics and optical Zermelo-Randers-Finsler geometry, Phys. Rev. D 79, 044022 (2009)
12. G.W. Gibbons, C.M. Warnick, The geometry of sound rays in a wind, Contemp. Phys. 52, 197 (2011)
13. G. Jannes, G. Rousseaux, The circular jump as a hydrodynamic white hole, ArXiv e-prints [arXiv:1203.6505 [gr-qc]] (2012)
14. R.K. Luneberg, Mathematical Theory of Optics (Brown University, Providence, RI, 1944)
15. J.C. Miñano, Perfect imaging in a homogeneous threedimensional region, Opt. Express 14, 9627 (2006)
16. J.D. Norton, The Hole Argument, in The Stanford Encyclopedia of Philosophy, ed. by E.N. Zalta, fall 2011 edn. (Stanford University, Stanford, CA, 2011). URL http://plato. stanford.edu/archives/fall2011/entries/spacetime-holearg/
17. I.I. Smolyaninov, Critical opalescence in hyperbolic metamaterials, J. Opt. 13, 125101 (2011)
18. G.W. Semenoff, Condensed-Matter Simulation of a Three-Dimensional Anomaly, Physical Review Letters 53, 2449 (1984)
19. M. Cvetič, G.W. Gibbons, Graphene and the zermelo optical metric of the BTZ black hole, Ann. Phys. (N.Y.) 327, 2617 (2012)
20. O. Boada, A. Celi, J.I. Latorre, M. Lewenstein, Dirac equation for cold atoms in artificial curved spacetimes, New J. Phys. 13, 035002 (2011)

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