# Tensor Operators in Loop Quantum Gravity

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**Abstract** We discuss how the quantization of the spinorial formalism for Loop Quantum Gravity naturally leads to the notion of tensor operators. These objects encode the natural structure to discuss observables associated to the intertwiner space. They allow in particular to deal with any type of gauge group, classical or quantum. After reviewing the standard case of SU(2), we focus on the specific example of $\mathscr{U}_q(\mathfrak{su}(2))$ and illustrate how dealing with a quantum group leads to the notion of quantum curved geometry.

## 1 Introduction

The current Loop Quantum Gravity (LQG) theory describes the quantum gravity regime with zero cosmological constant. The kinematical Hilbert space of the theory is spanned by quantum states for spatial geometries, the so-called *spin networks* [1]. Recently, it has been realized that spin network states are the quantization of some classical spinor states [2]. This spinorial formalism is reviewed in the next section, where we recall how it is linked to a discrete version of General Relativity, the *twisted geometries*. Then, focusing on a given vertex, we show how the spinorial structure associated to this vertex can be quantized in terms of tensor operators for the gauge group SU(2). This allows us to embed the U($N$) framework in a new mathematical formalism generalizable to the quantum group case [3].

In the second part, we show how the use of tensor operators in LQG can be generalized to the quantum group $\mathscr{U}_q(\mathfrak{su}(2))$. The use of a quantum group as gauge group

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instead of the Lie group SU(2) is motivated by the idea that this could be a way to introduce a non-vanishing cosmological constant $\Lambda$ in the LQG framework [4]. From tensor operators, we build observables for $\mathscr{U}_q(\mathfrak{su}(2))$-intertwiner that allow us to identify geometric observables for curved geometries such as the angle and length/area operators.

## 2 The spinorial formalism for LQG

The starting point of LQG is a smeared algebra, the holonomy-flux algebra, associated to graphs embedded into the spatial manifold. The continuum Ashtekar-Barbero variables are replaced by a pair $(g_e, X_e) \in \mathrm{SU}(2) \times \mathfrak{su}(2)$ associated to the edge $e$ of a given graph $\Gamma$.

At the quantum level, the kinematical Hilbert space of LQG is spanned by the spin network states, $|\Gamma, \{j_e\}, \{\iota_v\}\rangle$ where $j_e \in \mathbb{N}/2$ is a representation of SU(2) and is associated to each edge $e$ of $\Gamma$. $\iota_v$ is a SU(2)-intertwiner associated to the vertex $v$ of $\Gamma$. The geometrical interpretation of spin network states is provided by the properties of the angle, area and volume operators which are diagonalized by these quantum states. All geometric information is then encoded in the combinatorial aspects of the graphs.

Let us now focus on a given graph $\Gamma$ and consider a truncature of the full continuum theory to a finite Hilbert space $\mathscr{H}_\Gamma = L^2(\mathrm{SU}(2)^E, d^E g)$ with $E$ the number of edges of $\Gamma$ and $dg$ the Haar measure on SU(2). To understand what the classical degrees of freedom represented by the spin network states in $\mathscr{H}_\Gamma$ are, let us introduce classical *spinor networks*.

### 2.1 Classical spinor networks

We focus on a given edge $e$ of the graph $\Gamma$. This oriented edge is decorated by two spinors $|z_e\rangle = \begin{pmatrix} z_e^{(0)} \\ z_e^{(1)} \end{pmatrix} \in \mathbb{C}^2$ and $|\tilde{z}_e\rangle \in \mathbb{C}^2$ respectively associated to the source vertex $s(e)$ of $e$, and to the target vertex $t(e)$ of $e$. The phase space is defined by assuming that $|z_e\rangle$ is dual to its conjugate $|\tilde{z}_e\rangle$: $\{z_e^{(a)}, \bar{z}_e^{(b)}\} = -i\delta_{ab}$ where $a, b \in \{0, 1\}$. The space $\mathbb{C}^2 \times \mathbb{C}^2$, equipped with its canonical Poisson brackets, allows us to obtain the structure of the phase space of LQG on a given edge $e$, $\mathrm{SU}(2) \times \mathfrak{su}(2) \simeq T^*\mathrm{SU}(2)$. Indeed, the holonomy-flux algebra can be expressed in terms of the spinor variables[1]:

$$\mathbf{X}_e := \langle z_e | \sigma | z_e \rangle, \qquad g_e := \frac{|z_e\rangle[\tilde{z}_e| - |z_e]\langle\tilde{z}_e|}{\sqrt{\langle z_e | z_e\rangle\langle\tilde{z}_e|\tilde{z}_e\rangle}}, \tag{1}$$

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[1] We do not consider here the degenerate configuration $\langle z | z \rangle = 0$ which is equivalent to $|\mathbf{X}| = 0$. See [5] to see how this degenerate case can be treated.

with the additional area matching constraint on the spinors, $\mathscr{M}_e := \langle z_e | z_e \rangle - \langle \tilde{z}_e | \tilde{z}_e \rangle = 0$, where $\sigma^i$ are the Pauli matrices and $|z_e] := \begin{pmatrix} -\bar{z}_e^{(1)} \\ \bar{z}_e^{(0)} \end{pmatrix}$. Imposing this constraint $\mathscr{M}_e$ ensures that the two 3-vectors $\mathbf{X}_e$ and $\tilde{\mathbf{X}}_e$ have the same norm and that $g_e$ is unitary. The 6-dimensional space $T^*\mathrm{SU}(2)$ and its symplectic structure are recovered by symplectic reduction of $\mathbb{C}^2 \times \mathbb{C}^2$ by the constraint $\mathscr{M}_e$, $T^*\mathrm{SU}(2) \simeq (\mathbb{C}^2 \times \mathbb{C}^2)//\mathscr{M}$.

Let us now go back to the full graph $\Gamma$. The spinors are now labelled by a vertex $v$ and one of the edges $e$ which has $v$ for vertex. Thus, $\Gamma$ is decorated with a set of spinors $|z_{e,v}\rangle$. The components of the corresponding vectors $\mathbf{X}_{e,v}$ can be seen as generating $\mathrm{SU}(2)$ transformation on the spinor $|z_{e,v}\rangle$. The classical equivalent of the closure constraint which imposes the global $\mathrm{SU}(2)$ invariance at each node of the graph $\Gamma$ of a spin network state is simply written in terms of the 3-vectors as $\sum_{e \supset v} \mathbf{X}_{e,v} = \mathbf{0}$. This translates into a matricial constraint on the spinor variables, $\mathscr{C}_v \equiv \sum_{e \supset v} \left( |z_{e,v}\rangle\langle z_{e,v}| - \frac{1}{2}\langle z_{e,v}|z_{e,v}\rangle \mathbb{I} \right)$. If $E$ and $V$ denote respectively the number of edges of $\Gamma$ and the number of vertex of $\Gamma$, the symplectic reduction of $(\mathbb{C}^2 \times \mathbb{C}^2)//(\mathscr{M}_e)^E$ by $(\mathscr{C}_v)^V$ gives a symplectic space isomorphic to the gauge invariant phase-space of LQG on a fixed graph. Moreover, $\Gamma$ decorated by $\{|z_{e,v}\rangle / \mathscr{M}_e = 0$ and $\mathscr{C}_v = 0, \forall e, v \subset \Gamma\}$ defines a *spinor network*.

## 2.2 Twisted geometries

A nice geometrical interpretation of a spinor network comes from *twisted geometries* [6]. Essentially, a spinor network can be interpreted as a collection of polyhedra glued along their faces, where two shared faces have the same area but not necessary the same shape. The twisted geometry formalism is based on a seminal work by Minkowski which showed how given a vertex and a set of variables defining the twisted geometries, one can reconstruct a unique polyhedron dual to the vertex [7]. These variables and their relationship to the spinor variables are the following: $j_e \equiv \frac{\langle z_e | z_e \rangle}{2}$ the area of the dual surface to the edge $e$; two unit vectors $N_{e,s(e)}, \tilde{N}_{e,t(e)}$ such as $\mathbf{X}_{e,v}(z_{e,v}) = j_e N_{e,v}$; an angle $\xi_e \equiv -2(\arg(z_e^{(1)}) - \arg(\tilde{z}_e^{(1)}))$ the conjugate variable of $j_e$. $N_{e,s(e)}, \tilde{N}_{e,t(e)}$ are the two normals to the dual surface to the edge $e$ as seen from the two vertex frames sharing it. $\xi_e$ is related to the extrinsic curvature between the frames.

The name "twisted geometries" is motivated by this picture where geometries can be discontinuous at the faces connecting the polyhedra. This is related to the fact that the kinematical Hilbert space of LQG has room for torsion. In this sense, twisted geometries are more general than Regge Calculus which is torsion-free. For a graph with 4-valent vertices, Regge Calculus can be recovered by constraining the twisted geometries with some "shape-matching conditions" [8].

## 2.3 Tensor operators

The quantization of a spinor $|z\rangle$ and of its dual $|z]$ or more precisely of their conjugate variables give tensor operators of rank $1/2$ for SU(2):

$$\langle z| = (\bar{z}^{(0)}, \bar{z}^{(1)}) \rightarrow T^{1/2} = \begin{pmatrix} a^\dagger \\ b^\dagger \end{pmatrix}, \quad [z| = (-z^{(1)}, z^{(0)}) \rightarrow -\tilde{T}^{1/2} = \begin{pmatrix} -b \\ a \end{pmatrix}, \quad (2)$$

where $\alpha_i = a, b$ are harmonic oscillators, $[\alpha_i, \alpha_j^\dagger] = \delta_{ij}$, the other commutators being zero. More generally, a tensor operator of rank $j \in \mathbb{N}/2$, $\mathbf{t}_m^j$, is an object transforming as a vector $|j, m\rangle$ ($m \in \{-j, \cdots, +j\}$) under the adjoint action, $\triangleright$, of $\mathfrak{su}(2)$. We have $J_z \triangleright \mathbf{t}_m^j = [J_z, \mathbf{t}_m^j] = m\mathbf{t}_m^j$, $J_\pm \triangleright \mathbf{t}_m^j = [J_\pm, \mathbf{t}_m^j] = \sqrt{(j \mp m)(j \pm m + 1)}\, \mathbf{t}_{m \pm 1}^j$, where $J_z, J_\pm$ are the $\mathfrak{su}(2)$-generators.

Note that $J_z, J_\pm$ can also be seen as the components of a tensor operator of rank 1 for $\mathfrak{su}(2)$. And just as $|1, l\rangle = \sum_m C_{m\, l-m\, l}^{1/2\, 1/2\, 1} |1/2, m\rangle \otimes |1/2, m - l\rangle$ ($l \in \{-1, 0, 1\}$) where $C_{m\, m-l\, l}^{1/2\, 1/2\, 1}$ are Clebsch-Gordan (CG) coefficients, the tensor operator of rank 1 $(-J_+/\sqrt{2}, J_z, J_-/\sqrt{2})$ can be expressed in terms of tensor operators of ranks $1/2$ defined in (2) and we recover the Jordan-Schwinger representation of SU(2). The U($N$) formalism developed for $SU(2)$ intertwiners in terms of harmonic oscillators can also be rewritten in terms of tensor operators of rank $1/2$, $^{(i)}T^{1/2}$, $^{(i)}\tilde{T}^{1/2}$, where $i$ denotes the $i^{th}$ leg of a given vertex $v$. The observables $E_{ij} = a_i^\dagger a_j + b_i^\dagger b_j$, $F_{ij} = a_i b_j - a_j b_i$ and $F_{ij}^\dagger$ ($i, j \in \{1, \cdots, N\}$) for the space of $N$ valent SU(2)-intertwiners are simply rank 0 tensor operators built from these tensor operators of rank $1/2$ using CG coefficients to combine them into a scalar operator. These operators can be used to generate all observables for the intertwiner [9].

# 3 Generalization to $\mathscr{U}_q(\mathfrak{su}(2))$ as gauge group: towards curved discrete geometries?

## 3.1 Tensor operators

We focus on $\mathscr{U}_q(\mathfrak{su}(2))$ with $q$ real, which representation theory is similar to the one of $\mathfrak{su}(2)$. We refer to the Appendix for the definition and properties regarding $\mathscr{U}_q(\mathfrak{su}(2))$ [10]. As for the $q = 1$ case, a tensor operator is an operator $\mathbf{t}_m^j$ which transforms at the same time under the adjoint action, $\triangleright$, of $\mathscr{U}_q(\mathfrak{su}(2))$ and as the vector $|jm\rangle$,

$$K \triangleright \mathbf{t}_m^j = K\mathbf{t}_m^j K^{-1} = q^m \mathbf{t}_m^j \qquad (3)$$

$$J_\pm \triangleright \mathbf{t}_m^j = J_\pm\, \mathbf{t}_m^j\, K^{-1} - q^{\pm\frac{1}{2}} K^{-1}\, \mathbf{t}_m^j\, J_\pm = \sqrt{[j \mp m][j \pm m + 1]}\, \mathbf{t}_{m \pm 1}^j$$

The decomposition of a tensor operator into a product of tensor operators mentioned in the previous section remains valid. However, the *tensor product* of tensor operators becomes complicated to construct in the $q \neq 1$ case. Indeed, if $\mathbf{t}$ is a tensor operator then $^{(1)}\mathbf{t} = \mathbf{t} \otimes \mathbf{1}$ is a tensor operator, but $\mathbf{1} \otimes \mathbf{t}$ is in general *not* a tensor operator (it is however a tensor operator if $q = 1$, *i.e.* for $\mathfrak{su}(2)$). We need to use the $\mathscr{R}$-matrix to construct an intertwining map from the permutation. We say that the permutation composed with the $\mathscr{R}$-matrix is a $q$-deformed permutation. Starting from a given $\mathbf{t}$ of rank $j$, we can build $N$ tensor operators of rank $j$ using consecutive deformed permutations. For all $i \in \{1, \cdots, N\}$,

$$^{(i)}\mathbf{t} = (\mathscr{R}_{ii-1}..\mathscr{R}_{i1}(\mathbf{1} \otimes ... \otimes \mathbf{t})\mathscr{R}_{i1}^{-1}..\mathscr{R}_{ii-1}^{-1}) \otimes \mathbf{1} \otimes ..\mathbf{1} \tag{4}$$

is a tensor operator of same rank as $\mathbf{t}$. The fundamental tensor operators are the tensor operators of rank $1/2$, the *spinor* operators. Similarly to the $q = 1$ case, a pair of $q$-harmonic oscillators provides a convenient set of variables to realize these operators [11]. The annihilation and creation operators $\alpha_i = a_i, b_i$, $\alpha_i^\dagger = a_i^\dagger, b_i^\dagger$, and the number operator $N_{\alpha_i}$ satisfy now the following conditions

$$[\alpha_i, \alpha_j] = [\alpha_i^\dagger, \alpha_j^\dagger] = 0, \quad [\alpha_i, \alpha_j^\dagger]_{q^{\pm \frac{1}{2}}} = \delta_{ij} q^{\mp N_{\alpha_i}}, \tag{5}$$

$$q^{N_{\alpha_i}/2} \alpha_i^\dagger = q^{1/2} \alpha_i^\dagger q^{N_{\alpha_i}/2}, \quad q^{N_{\alpha_i}/2} \alpha_i = q^{-1/2} \alpha_i q^{N_{\alpha_i}/2}.$$

We can use this pair of harmonic oscillators to construct the Jordan-Schwinger realization the $\mathscr{U}_q(\mathfrak{su}(2))$ generators [11]

$$J_z = \frac{1}{2}(N_a - N_b), \quad J_+ = a^\dagger b, \quad J_- = b^\dagger a, \tag{6}$$

which are *not* the components of a tensor operator of rank 1, contrary to the $q = 1$ case. Using (5), we can recover the $\mathscr{U}_q(\mathfrak{su}(2))$ commutation relations provided in (10). We can also use the Fock space of this pair $q$-harmonic oscillators to generate the representations of $\mathscr{U}_q(\mathfrak{su}(2))$.

Thanks to this realization, we can find the two solutions of (3) for $j = 1/2$, which transform therefore as spinors.

$$\mathbf{t}^{\frac{1}{2}} = \begin{pmatrix} a^\dagger q^{N_a/4} \\ b^\dagger q^{(2N_a+N_b)/4} \end{pmatrix} \xrightarrow{q \to 1} T^{1/2}, \; \tilde{\mathbf{t}}^{\frac{1}{2}} = \begin{pmatrix} q^{(2N_a+N_b+1)/4} b \\ -q^{(N_a-1)/4} a \end{pmatrix} \xrightarrow{q \to 1} -\tilde{T}^{1/2}. \tag{7}$$

Using the relevant CG coefficients, we construct the operators $\mathbf{t}^1$ which transform as vectors, $\mathbf{t}^1_{\pm 1} = \mathbf{t}^{\frac{1}{2}}_\pm \tilde{\mathbf{t}}^{\frac{1}{2}}_\pm$, $\mathbf{t}^1_0 = \frac{1}{\sqrt{[2]}} \left( q^{-\frac{1}{4}} \mathbf{t}^{\frac{1}{2}}_+ \tilde{\mathbf{t}}^{\frac{1}{2}}_- + q^{\frac{1}{4}} \mathbf{t}^{\frac{1}{2}}_- \tilde{\mathbf{t}}^{\frac{1}{2}}_+ \right)$. Explicitly, we have,

$$\mathbf{t}^1_{\pm 1} = \mp \frac{q^{-\frac{1}{2}}}{\sqrt{[2]}} q^{\frac{1}{2}(N_a+N_b)} q^{\frac{J_z}{2}} J_\pm, \xrightarrow{q \to 1} \mp \frac{J_\pm}{\sqrt{2}} \tag{8}$$

$$\mathbf{t}^1_0 = \frac{q^{-\frac{1}{2}}}{[2]} q^{\frac{1}{2}(N_a+N_b)} (q^{-1/2} J_+ J_- - q^{1/2} J_- J_+) \xrightarrow{q \to 1} J_z.$$

Any other tensor operator of rank $j$ can be built in a similar way by combining spinor operators and CG coefficients. Then, the construction of tensor operators of rank $j$ from tensor products of a given tensor operator can be done using (4). Note that in general $^{(n)}\mathbf{t}^{j_1}$ and $^{(m)}\tilde{\mathbf{t}}^{j_2}$ will not commute in the quantum group case, contrary to the $q = 1$ case.

### 3.2 Observables for the $q$-deformed intertwiner space

In LQG, the intertwiner $|\iota_{j_1..j_N}\rangle$ is understood as the fundamental chunk of quantum space. We have seen in the previous section that the use of tensor operators allows to construct operators invariant under the adjoint action of $\mathfrak{su}(2)$ and to recover the complete algebra of observables defined in the U($N$) framework. The tensor operator formalism allows us to extend this framework to the quantum group case in a direct manner.

As in the $\mathfrak{su}(2)$ case, each leg $k$ of the intertwiner corresponds to a representation $V^{j_k}$. We can associate with each leg a tensor operator $^{(k)}\mathbf{t}^{\frac{1}{2}}$. Using $\mathscr{U}_q(\mathfrak{su}(2))$ recoupling theory, it is possible to build from these tensor operators a tensor operator of rank 0, $i.e.$ an $observable$. Let us denote $|\iota^q_{j_1..j_N}\rangle$ the intertwiner defined from representations of $\mathscr{U}_q(\mathfrak{su}(2))$. Using spinor operators, as in the $q = 1$ case, there are only three types of observables that can be constructed[2].

$$\mathscr{E}_{ij} = -\sqrt{[2]} \sum_{m_i} {}_q\mathbf{C}\,{}^{\frac{1}{2}\ \frac{1}{2}\ 0}_{m_1 m_2 0}\,{}^{(i)}\mathbf{t}^{\frac{1}{2}}_{m_1}\,{}^{(j)}\tilde{\mathbf{t}}^{\frac{1}{2}}_{m_2} \xrightarrow{q \to 1} a^\dagger_i a_j + b^\dagger_i b_j = E_{ij}$$

$$\mathscr{G}^\dagger_{ij} = -\sqrt{[2]} \sum_{m_i} {}_q\mathbf{C}\,{}^{\frac{1}{2}\ \frac{1}{2}\ 0}_{m_1 m_2 0}\,{}^{(i)}\mathbf{t}^{\frac{1}{2}}_{m_1}\,{}^{(j)}\mathbf{t}^{\frac{1}{2}}_{m_2} \xrightarrow{q \to} a^\dagger_i b^\dagger_j - b^\dagger_i a^\dagger_j = F^\dagger_{ij}$$

$$\mathscr{F}_{ij} = -\sqrt{[2]} \sum_{m_i} {}_q\mathbf{C}\,{}^{\frac{1}{2}\ \frac{1}{2}\ 0}_{m_1 m_2 0}\,{}^{(i)}\tilde{\mathbf{t}}^{\frac{1}{2}}_{m_1}\,{}^{(j)}\tilde{\mathbf{t}}^{\frac{1}{2}}_{m_2} \xrightarrow{q \to 1} a_i b_j - b_i a_j = F_{ij},$$

where the operators $E_{ij}$, $F^\dagger_{ij}$, $F_{ij}$ of the $U(N)$ formalism are recovered. Preliminary results indicate that the $\mathscr{E}_{ij}$ can be expressed as functions of the generators of $\mathscr{U}_q(\mathfrak{u}(N))$ [3]. This means that the $\mathscr{U}_q(\mathfrak{su}(2))$ intertwiner can be seen as a representation of $\mathscr{U}_q(\mathfrak{u}(N))$, a natural generalization of the classical case, where the $E$'s form a $\mathfrak{u}(N)$ algebra.

Using vector operators we can construct different interesting observables. Considering $^{(i)}\mathbf{t}^1$ and $^{(j)}\mathbf{t}^1$, we construct the following scalar operator, $^{(i)}\mathbf{t}^1 \cdot {}^{(j)}\mathbf{t}^1 \equiv {}_q\mathbf{C}\,{}^{1\ 1\ 0}_{m_1 m_2 0}\,{}^{(i)}\mathbf{t}^1_{m_1}\,{}^{(j)}\mathbf{t}^1_{m_2}$. Using recoupling theory, we calculate the action of this operator on a three legs intertwiner $|\iota^q_{j_b j_c j_a}\rangle$. For simplicity, we assume now a 2d space, ie LQG for a 2+1 spacetime. We can then interpret this intertwiner as the quantum

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[2] The other ordering choice for $^{(i)}\mathbf{t}^{\frac{1}{2}}_{m_1}$ and $^{(j)}\tilde{\mathbf{t}}^{\frac{1}{2}}_{m_2}$ is equivalent to our choice modulo a rescaling of the operators.

state of a triangle. In the limit $q \to 1$, we know that $^{(b)}\mathbf{t}^1 \cdot {}^{(c)}\mathbf{t}^1 \propto {}^{(b)}\mathbf{J} \cdot {}^{(c)}\mathbf{J}$ is interpreted as the quantization of the cosine of the angle $\theta_a$ between the tangent vectors $\hat{u}_c$, $\hat{u}_b$ (cf Fig. 1).
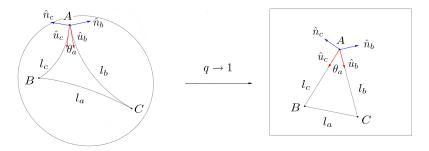


**Fig. 1** Hyperbolic and flat triangles represented in the Poincaré disk and the plane respectively.

The action of $^{(b)}\mathbf{J} \cdot {}^{(c)}\mathbf{J}$ on $|\iota_{j_b j_c j_a}^{q=1}\rangle$ leads to a quantized version of the flat cosine law [12]. When performing the calculation with $q \neq 1$, we obtain that the action of $^{(b)}\mathbf{t}^1 \cdot {}^{(c)}\mathbf{t}^1$, for an appropriate choice of normalization, is diagonal on $|\iota_{j_b j_c j_a}^{q}\rangle$ with eigenvalue,

$$\frac{\cosh\frac{\lambda}{2}\cosh((j_a + \frac{1}{2})\lambda) - \cosh((j_b + \frac{1}{2})\lambda)\cosh((j_c + \frac{1}{2})\lambda))}{\sinh((j_b + \frac{1}{2})\lambda)\sinh((j_c + \frac{1}{2})\lambda)}, \tag{9}$$

where $\lambda = \frac{l_p}{R}$ with $l_p$ the Planck length, $R$ the cosmological radius, $R = \frac{1}{\sqrt{\Lambda}}$ and $\Lambda$ the positive cosmological constant. This suggest now that we are dealing with a quantization of the *hyperbolic cosine law*.

$$-\hat{n}_b \cdot \hat{n}_c = \hat{u}_b \cdot \hat{u}_c = \cos\theta_a = \frac{-\cosh\frac{l_a}{R} + \cosh\frac{l_b}{R}\cosh\frac{l_c}{R}}{\sinh\frac{l_b}{R}\sinh\frac{l_c}{R}}.$$

When $R$ goes to infinity, we recover the standard flat cosine law. This strongly suggests that dealing with $\mathscr{U}_q(\mathfrak{su}(2))$ 3-legs intertwiner encodes a quantization of the curved hyperbolic triangle (cf Fig. 1), where the quantized length is $\hat{l}_i = j_i + \frac{1}{2}$. Morevoer, if we consider the case where $j_a = 0$ and $j_b$, $j_c \neq 0$, unlike the classical case, we do not get $\theta_a = 0$. This means that the introduction of curvature induced a minimum angle as already guessed in [13].

A similar argument can be made when dealing with 3d spatial geometries. In this case, $\lambda = \frac{l_p^2}{R^2}$. The vectors operators encode the quantization of the normals to surfaces. The area operator is then quantized with eigenvalues $j + \frac{1}{2}$, and not as the $\mathscr{U}_q(\mathfrak{su}(2))$ Casimir as usually proposed in the quantum group case [4].

To recover a quantization of the hyperbolic cosine law shows that we are on the right track to determine the variables encoding a curved twisted geometry. The natural candidates are the complex variables $z_i$ variables which are the "classical" analogues of the $q$-harmonic oscillators (5), equipped with a $q$-calculus. In this case, the relevant differential is given by $D^q f(z) \equiv (f(zq^{\frac{1}{2}}) - f(zq^{-\frac{1}{2}}))/(z(q^{\frac{1}{2}} - q^{-\frac{1}{2}}))$. We leave this for further investigations.

## 4 Outlook

Twisted geometries and the LQG spinorial reformulation provide an interesting framework to get a better understanding of the classical degrees of freedom carried by a spin network, a quantum state of space. When quantizing this spinorial formalism of LQG, tensor operators arise naturally. These objects are easy to generalize to the quantum group case. Whereas at this stage, we do not know clearly how to introduce the cosmological constant in the LQG framework, we have showed how using the quantum group does encode the presence of a cosmological constant in the geometry dual to an intertwiner. In particular we have obtained a quantization of the hyperbolic cosine law.

Morevoer, identifying these tensor operators as the right building blocks provides new techniques to eventually solve the issue of the quantum groups appearance in the LQG context as well as new directions to explore.

For example, it is now possible using the $\mathscr{E}$'s and $\mathscr{F}$'s observables to generalize the Hamiltonian constraint for 3d gravity proposed in [14] in order to get an invariant Hamiltonian operator under $\mathscr{U}_q(\mathfrak{su}(2))$. This would allow to connect for the first time an LQG Hamiltonian constraint and the $6j$-symbols for $\mathscr{U}_q(\mathfrak{su}(2))$ which appears in the Turaev-Viro model. This would allow to probe how the use of quantum group is a good approximation for encoding the cosmological constant.

## Appendix: $\mathscr{U}_q(\mathfrak{su}(2))$

The quasi-triangular Hopf algebra $\mathscr{U}_q(\mathfrak{su}(2))$ is generated by the elements $J_\pm, K = q^{\frac{J_z}{2}}$ which satisfy

$$KJ_\pm K^{-1} = q^{\pm\frac{1}{2}}J_\pm, \quad [J_+, J_-] = \frac{K^2 - K^{-2}}{q^{1/2} - q^{-1/2}}. \tag{10}$$

The coproduct $\Delta : \mathscr{U}_q(\mathfrak{su}(2)) \to \mathscr{U}_q(\mathfrak{su}(2)) \otimes \mathscr{U}_q(\mathfrak{su}(2))$ and antipode $S : \mathscr{U}_q(\mathfrak{su}(2)) \to \mathscr{U}_q(\mathfrak{su}(2))$ are given by

$$\Delta K = K \otimes K, \ \Delta J_\pm = J_\pm \otimes K + K^{-1} \otimes J_\pm, \ SK = K^{-1}, \ SJ_\pm = -q^{\pm 1/2}J_\pm.$$

The $\mathscr{R}$-matrix $\mathscr{R} \in \mathscr{U}_q(\mathfrak{su}(2)) \otimes \mathscr{U}_q(\mathfrak{su}(2))$ encodes the quasi-triangular structure, which tells us how much the coproduct is non-commutative. If we note $\psi : \mathscr{U}_q(\mathfrak{su}(2)) \otimes \mathscr{U}_q(\mathfrak{su}(2)) \to \mathscr{U}_q(\mathfrak{su}(2)) \otimes \mathscr{U}_q(\mathfrak{su}(2))$ the permutation, then we have $\psi \circ \Delta X = \mathscr{R}(\Delta X)\mathscr{R}^{-1}$, $\forall X \in \mathscr{U}_q(\mathfrak{su}(2))$. Standard notations are $\mathscr{R}_{12} = \sum R_1 \otimes R_2$, $\mathscr{R}_{21} = \sum R_2 \otimes R_1$, ... When $q$ is real, the representation theory of $\mathscr{U}_q(\mathfrak{su}(2))$ is essentially the same as that of $\mathfrak{su}(2)$ [10]. A representation $V^j$ is hence generated by the vectors $|j,m\rangle$ with $j \in \mathbb{N}/2$ and $m \in \{-j,..,j\}$. The key-difference is that we use $q$-numbers $[x] \equiv \frac{q^{x/2}-q^{-x/2}}{q^{1/2}-q^{-1/2}}$.

$$K|j,m\rangle = q^m|j,m\rangle, \quad J_\pm|j,m\rangle = \sqrt{[j\mp m][j\pm m+1]}\,|j,m\pm 1\rangle.$$

The adjoint action of $\mathscr{U}_q(\mathfrak{su}(2))$ on some operator $\mathscr{O}$ is

$$J_\pm \triangleright \mathscr{O} = J_\pm \mathscr{O} K^{-1} - q^{\pm 1/2} K^{-1} \mathscr{O} J_\pm, \; K \triangleright \mathscr{O} = K \mathscr{O} K^{-1}.$$

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