Analytical Conformal Compactification of Schwarzschild Spacetime

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Abstract We discuss a construction of the coordinates simultaneously covering the complete Schwarzschild manifold as well as its conformal extension beyond $\mathscr{I}^{\pm}$. We provide an example of such coordinates and show they are analytical both at horizon and at null infinity. We also show, that having such analytical compactification can improve convergence in certain numerical applications.

The Penrose–Carter diagrams became a standard way to visualize various aspects of geometrical objects and physical processes in black-hole spacetimes. The most widely used prescription to compactify the Schwarzschild coordinates comes from the textbook [1]. It is known it depicts the regions near infinity dissimilarly to those of compactified Minkowski spacetime and thus more recent textbooks present modified transformations from Kruskal to compactified coordinates [2, 3]. Unfortunately, even these transformations do not provide analytical coordinates at null infinites $\mathscr{I}^{\pm}$.

Spacetimes which at distant regions resemble the Minkowski spacetime form a class of *asymptotically flat spacetimes* (*AFS*) [4, 5]. For such spacetimes coordinates and appropriate conformal factor Ω exist that make the conformally related metric

$$\widetilde{ds}^2 = \Omega^2\, ds^2 \tag{1}$$

regular at null infinity, where the conformal factor must vanish at infinity and the leading terms of its expansion near $\mathscr{I}$ are prescribed

$$\Omega(\mathscr{I}^{\pm}) = 0, \qquad \widetilde{\nabla}_\mu \Omega(\mathscr{I}^{\pm}) \neq 0, \tag{2}$$

$$\Omega(i^0) = 0, \qquad \widetilde{\nabla}_\mu \Omega(i^0) = 0, \qquad \widetilde{\nabla}_\mu \widetilde{\nabla}_\nu \Omega(i^0) = 2\widetilde{g}_{\mu\nu}(i^0). \tag{3}$$

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We construct an explicit transformation and a conformal factor satisfying these conditions for the Schwarzschild spacetime with

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\omega^2. \tag{4}$$

We generalize the coordinates smoothly covering $\mathscr{I}^\pm$ given in [4] in the form of a direct transformation between Schwarzschild coordinates t, r and the compactified null coordinates u, v

$$f(r(u,v)) = h(v) + h(-u), \tag{5}$$

$$t(u,v) = h(v) - h(-u). \tag{6}$$

Here $f(r) = r + 2M\ln[r/(2M) - 1]$ denotes Regge–Wheeler tortoise coordinate. We put the horizon at $u = 0$ and $v = 0$, the past null infinity $\mathscr{I}^-$ at $u = -\pi/2$, and $\mathscr{I}^+$ at $v = \pi/2$. If the function h is written as a combination of two analytic functions $h(z) = \alpha(z) + 2M\ln\beta(z)$, then using the usual conformal factor $\Omega \sim \cos u \cos v$ we can show, that the conformally related metric (1) obtained by such transformation is analytic both at the horizon and at $\mathscr{I}^\pm$ if certain behavior of the functions α and β at $z = 0$ and $z = \pm\pi/2$ is satisfied.

A careful analysis shows that the function

$$h(x) = \frac{M}{\cos x} + 2M\ln\frac{1 - \cos x}{\sin x \, \cos x} \tag{7}$$

leads to the transformation (5-6) which provides analytic coverage of all regions of Kruskal's complete manifold as well as of the regions beyond null infinites of the conformally related manifold. The compactified line element (1) can be then simplified into

$$\widetilde{\mathrm{d}s}^2 = \frac{1 - \dfrac{2M}{r(u,v)}}{4M^2 \sin u \sin v}\,\mathrm{d}u\,\mathrm{d}v + \Omega^2\, r(u,v)^2 \mathrm{d}\omega^2, \tag{8}$$

where we have absorbed the derivatives of h into the conformal factor

$$\Omega(u,v) = \frac{\cos u \cos v}{4M^2\sqrt{(1+\cos u)^2 - 2\cos^3 u}\sqrt{(1+\cos v)^2 - 2\cos^3 v}}. \tag{9}$$

Please note, that no factors similar to $\exp(-r/2M)$ which in Kruskal's coordinates spoil the behavior at null infinities appear in (8).

In Fig. 1 we illustrate the way the coordinates t and r cover the Penrose–Carter diagram of compactified Schwarzschild spacetime, namely we can see how these coordinates behave near spatial infinity i^0, the way the singularity $r = 0$ and event horizon $r = 2M$ meet at future time-like infinity i^+ and that a region of negative r appears behind $\mathscr{I}^\pm$. The fact that the conformal metric is an analytical function of compactified coordinates has to be be proven from mathematical properties of functions which appear in (5)-(9) – the most complicated is the proof of analyticity at $\mathscr{I}^{\pm}$, where we have to use either the implicit function theorem or theorem on properties of solution of ordinary differential equations.

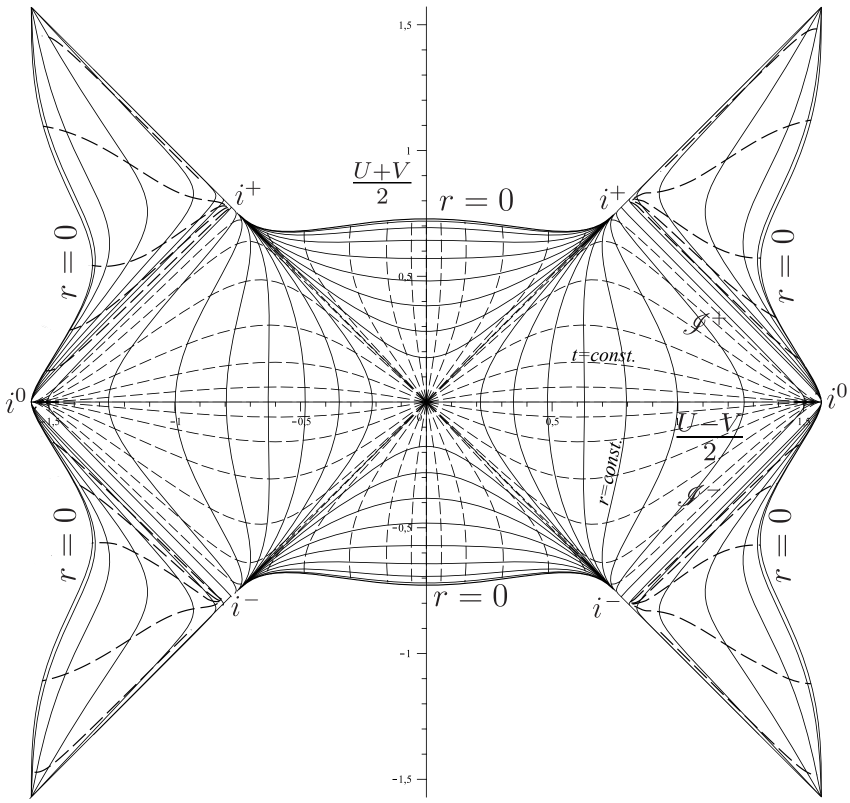

Fig. 1 The Carter-Penrose diagram of the Schwarzschild spacetime covering its analytical extension beyond $\mathscr{I}^{\pm}$.

To stress the importance of analytic coordinates we compare the coordinates (5-6) with those suggested in [2] in a simple test. We assume a situation when a problem is formulated as a differential equation which is numerically solved on the compactified Schwarzschild spacetime. Coefficients in this equation reflect the curved geometry of the spacetime and typically contain function $1/r = \Omega\, \tilde{g}_{\theta\theta}^{-1/2}$. This function inherits analytical properties of $\tilde{g}_{\mu\nu}$ and Ω. In this test we consider a slice (of hyperboloidal type) which spans from the horizon to the null infinity with parameter $s \in [0,1]$ determining coordinates $u = -s\pi/4, v = (1+s)\pi/4$.

In Fig. 2 (left) we plot function $1/r(s)$ along the slice. Indeed, we cannot distinguish which function behaves better. We also decompose both functions into Chebyshev series and in Fig. 2 (right) plot absolute values of the coefficients showing that for analytic compactification the coefficients decay much faster (exponentially). Since typically numerical methods of solution of differential equations work better when coefficients in the differential equation are analytical functions, it seems that even for problems which happen entirely in the physical domain of the Schwarzschild manifold the way the coordinates pierce through $\mathscr{I}^{\pm}$ matters.

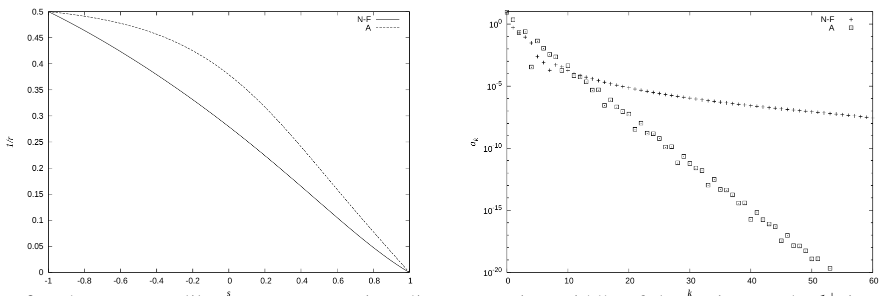

Fig. 2 When a space-like curve representing slice connecting middle of the horizon and $\mathscr{I}^{+}$ in Penrose–Carter diagram is linearly parametrized straight line with the parameter $s \in (-1,1)$, function $1/r$ at Novikov's-Frolov's and analytical coordinates can be plotted as a function of this parameter s.

We found a new way of constructing analytical coordinates for *AFS* and showed a related diagram for the Schwarzschild spacetime. The same method can be used for other vacuum *AFSs*, i.e. Reissner–Nordström or extreme Reissner–Nordström. We demonstrated advantages of using this coordinate system for numerical methods in general relativity.

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