Unruh-DeWitt Detector on the BTZ Black Hole

Lee Hodgkinson and Jorma Louko

Abstract We examine an Unruh-DeWitt particle detector coupled to a scalar field in three-dimensional curved spacetime, within first-order perturbation theory. We first obtain a causal and manifestly regular expression for the instantaneous transition rate in an arbitrary Hadamard state. We then specialise to the Bañados-Teitelboim-Zanelli black hole and to a massless conformally coupled field in the Hartle-Hawking vacuum. A co-rotating detector responds thermally in the expected local Hawking temperature, while a freely-falling detector shows no evidence of thermality in regimes that we are able to probe, not even far from the horizon. The boundary condition at the asymptotically anti-de Sitter infinity has a significant effect on the transition rate.

1 Introduction

Whenever non-inertial observers or curved backgrounds are present in quantum field theory, the notions of vacuum state and particle number become non-unique. For this reason it proves convenient to define particles operationally; that is to say, we couple the field to a simple quantum mechanical system that we think of as our detector and define particles via the field's interaction with the energy levels of this system. Upwards (respectively downwards) transitions can be interpreted as due to absorption (emission) of field quanta, or particles. This is the Unruh-DeWitt model for a particle detector [1, 2].

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Lee Hodgkinson and Jorma Louko
School of Mathematical Sciences, University of Nottingham, Nottingham NG7 2RD, UK
e-mail: pmxlh1@nottingham.ac.uk

Jorma Louko
Kavli Institute for Theoretical Physics, University of California, Santa Barbara, CA 93106-4030, USA, e-mail: jorma.louko@nottingham.ac.uk

In this contribution we address a pointlike Unruh-DeWitt detector coupled to a scalar field in three-dimensional spacetime, within first-order perturbation theory. We first find the detector's instantaneous transition rate in an arbitrary Hadamard state. We then specialise to a massless conformally coupled field on the Bañdados-Teitelboim-Zanelli (BTZ) black hole, in the Hartle-Hawking vacuum, analysing the thermal response seen by a co-rotating detector and the time evolution of the response of a freely-falling detector. A longer exposition of the results can be found in [3].

2 Transition rate in $(2+1)$ dimensions

With the Unruh-DeWitt detector, the fundamental quantity of interest is the probability of a transition between the energy eigenstates. In the framework of first order perturbation theory the probability for a transition of energy E is proportional to the response function,

$$\mathscr{F}(E) = 2 \lim_{\varepsilon \to 0_+} \Re \int_{-\infty}^{\infty} du \, \chi(u) \int_0^{\infty} ds \, \chi(u-s) \, e^{-iEs} W_\varepsilon(u, u-s), \qquad (1)$$

where χ is a smooth switching function that turns on (off) the detector's interaction with the field and $W_\varepsilon(u, u-s)$ is a one-parameter family of functions that converge to the pull-back of the Wightman distribution to the detector's wordline [4, 5, 6, 7]. A related quantity of interest is the transition rate, which can be defined as the derivative of the transition probability with respect to the total detection time. One must take great care when obtaining the transition rate from the response function [8, 9, 10, 11]. We will adopt the approach developed in [12, 7, 13] of taking a controlled sharp switching limit.

In three-dimensional spacetime, the Wightman distribution $W(\mathsf{x}, \mathsf{x}')$ of a real scalar field in a Hadamard state can be represented by the $\varepsilon \to 0_+$ limit of a family of functions with the short distance form [14]

$$W_\varepsilon(\mathsf{x}, \mathsf{x}') = \frac{1}{4\pi} \left[\frac{U(\mathsf{x}, \mathsf{x}')}{\sqrt{\sigma_\varepsilon(\mathsf{x}, \mathsf{x}')}} + \frac{H(\mathsf{x}, \mathsf{x}')}{\sqrt{2}} \right], \qquad (2)$$

where ε is a positive parameter, $\sigma(\mathsf{x}, \mathsf{x}')$ is the squared geodesic distance between x and x', $\sigma_\varepsilon(\mathsf{x}, \mathsf{x}') := \sigma(\mathsf{x}, \mathsf{x}') + 2i\varepsilon [T(\mathsf{x}) - T(\mathsf{x}')] + \varepsilon^2$ and T is any globally-defined future-increasing C^∞ function. The branch of the square root is such that the $\varepsilon \to 0_+$ limit of the square root is positive when $\sigma(\mathsf{x}, \mathsf{x}') > 0$ [14, 6]. Here $U(\mathsf{x}, \mathsf{x}')$ and $H(\mathsf{x}, \mathsf{x}')$ are symmetric biscalars which have expansions governed by certain recursion relations [14], and they are regular in the coincidence limit.

Given (2), the detector's instantaneous transition rate can be shown to take the form [3]

$$\hat{\mathscr{F}}_\tau(E) = \frac{1}{4} + 2\int_0^{\tau-\tau_0} ds \, \Re \left[e^{-iEs} W_0(\tau, \tau-s) \right] , \qquad (3)$$

where τ_0 is the proper time at which the detector was switched on, τ is the proper time at which the instantaneous transition rate is read off, and the function W_0 is the pointwise $\varepsilon \to 0_+$ limit of W_ε. We are assuming that any singularities that $W(\mathsf{x}, \mathsf{x}')$ may have at $\sigma(\mathsf{x}, \mathsf{x}') \neq 0$, not captured by the asymptotic expansion (2), are so mild that taking the pointwise limit is valid. Such singularities will in particular occur in the BTZ spacetime below.

3 Detector in the BTZ spacetime

We now specialise to a detector in the BTZ black hole spacetime [15, 16, 17]. This spacetime can be obtained by periodically identifying AdS$_3$, and in coordinates adapted to the global isometries the metric takes the form

$$ds^2 = -(N^\perp)^2 dt^2 + f^{-2}dr^2 + r^2\left(d\phi + N^\phi dt\right)^2 , \qquad (4)$$

where $N^\perp = f = \left(-M + \frac{r^2}{\ell^2} + \frac{J^2}{4r^2}\right)^{1/2}$, $N^\phi = -\frac{J}{2r^2}$, ℓ is a positive parameter that sets the AdS$_3$ curvature scale, ϕ has period 2π, and a non-extremal black hole is obtained when the mass parameter M and the angular momentum parameter J satisfy $|J| < M\ell$. The spacetime has many similarities with the Kerr black hole, but its null infinities are asymptotically AdS, as opposed to asymptotically flat. The conformal diagram of the $J = 0$ case is shown in Figure 1. The importance of this asymptotic structure for us is that the spacetime is not globally hyperbolic, and to build a sensible quantum field theory one must impose boundary conditions at the infinity. We shall see that the detector response turns out to be highly sensitive to these boundary conditions.
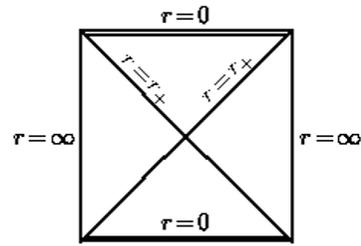
Fig. 1 Conformal diagram for the $J = 0$ BTZ black hole. The Killing horizon of the Killing vector ∂_t is at $r = r_+$, where $r_+ = \sqrt{M}\,\ell$.

We consider a massless, conformally coupled field. We first introduce on the covering space AdS$_3$ the three AdS-invariant states whose Wightman functions are given by [17]

$$G_A^{(\zeta)}(\mathsf{x},\mathsf{x}') = \frac{1}{4\pi} \left(\frac{1}{\sqrt{\Delta\mathsf{X}^2(\mathsf{x},\mathsf{x}')}} - \frac{\zeta}{\sqrt{\Delta\mathsf{X}^2(\mathsf{x},\mathsf{x}') + 4\ell^2}} \right), \qquad (5)$$

where $\Delta\mathsf{X}^2(\mathsf{x},\mathsf{x}')$ is the squared geodesic distance between x and x' in the flat $\mathbb{R}^{2,2}$ in which AdS_3 can be embedded as a submanifold, the parameter $\zeta \in \{0, 1, -1\}$ specifies whether the boundary condition at infinity is respectively transparent, Dirichlet or Neumann, and we have suppressed the $i\varepsilon$ that controls the short distance form (2). The Wightman function in the induced state on the BTZ spacetime is then given by the image sum [17]

$$G_{\mathrm{BTZ}}(\mathsf{x},\mathsf{x}') = \textstyle\sum_n G_A(\mathsf{x}, \Lambda^n \mathsf{x}'), \qquad (6)$$

where $\Lambda\mathsf{x}'$ denotes the action on x' of the isometry $(t, r, \phi) \mapsto (t, r, \phi + 2\pi)$, and the notation suppresses the distinction between points on AdS_3 and points on the quotient spacetime. The scalar field is assumed untwisted so that no additional phase factors appear in (6).

The detector's transition rate is obtained by substituting (6) into (3). In sections 4 and 5 we discuss the transition rate for selected detector trajectories.

4 Co-rotating detector in BTZ

As the first example we consider a detector that is in the exterior region of the BTZ black hole, at constant value of r and co-rotating with the horizon angular velocity Ω_H. In the special case $J = 0$, we have $\Omega_H = 0$ and the detector is static. Unlike in Kerr, these detector trajectories exist at arbitrarily large values of r: this is a consequence of the AdS asymptotics.

As the detector is stationary, we take the switch-on to be in the asymptotic past. The Wightman function turns out to contain singularities between timelike-separated points on the detector's trajectory, but the consequent singularities in the transition rate formula (3) are integrable and the transition rate remains well defined. Further, contour manipulations allow the transition rate to be cast in a manifestly nonsingular form that is amenable to analytic techniques, including asymptotic analyses in a number of asymptotic regimes, as well as to numerical evaluation. We can in particular verify analytically that the transition rate satisfies

$$\dot{\mathscr{F}}(E) = \mathrm{e}^{-E/T_{\mathrm{loc}}} \dot{\mathscr{F}}(-E), \qquad (7)$$

where T_{loc} is the co-rotating Hawking temperature at the detector's location [17]. (As the transition rate is stationary, we have dropped the subscript τ.) The transition rate is hence thermal in the local Hawking temperature in the sense of the Kubo-Martin-Schwinger (KMS) property [18, 19], as expected from the general properties of the Hartle-Hawking vacuum [20, 21].

The boundary condition at the infinity is found to have a significant effect on the quantitative properties of the transition rate. The special case of a spinless black

hole, with a detector at large and small distances from the hole, is illustrated in Figure 2.
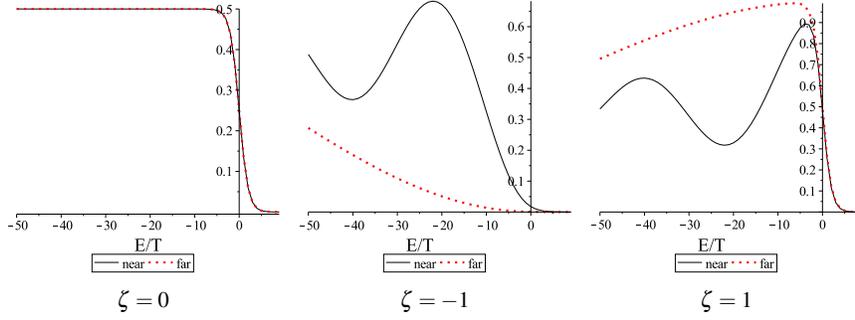
Fig. 2 $\dot{\mathscr{F}}$ for a co-rotating detector, as a function of the detector's energy gap E divided by the local Hawking temperature T, for a large non-spinning hole, with the detector near the hole (solid) and far from hole (dotted). Note the significant differences between the three boundary conditions.
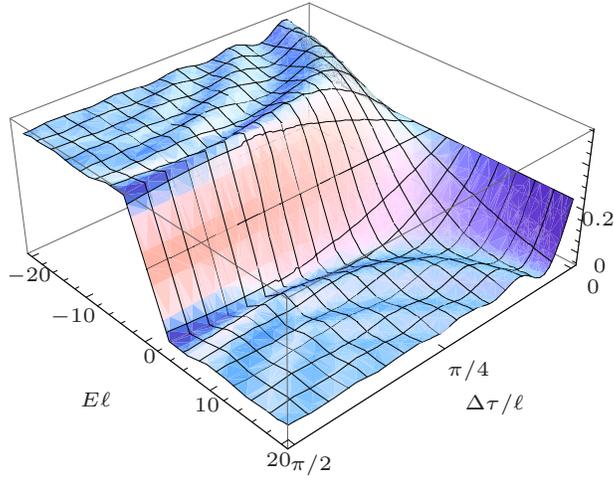
Fig. 3 The transition rate of a detector on a radial geodesic in the spinless BTZ spacetime, assuming that the mass is large and that the switch-on and switch-off moments are not close to the white hole and black hole singularities, with the transparent boundary condition at the infinity. The horizontal axes are the detector's energy gap E and the total detection time $\Delta\tau := \tau - \tau_0$, normalised by the AdS scale ℓ. Note the dominance of the de-excitation rate ($E < 0$) over the excitation rate ($E > 0$) after the transient switch-on effects have died out.

5 Inertial detector in BTZ

As the second example we consider a detector on a geodesic that falls radially into the spinless black hole. This trajectory is not stationary and the transition rate depends on both the switch-on moment and the switch-off moment. Furthermore the switch-on moment cannot be pushed to the infinite past because the trajectory starts at the white hole singularity at a finite proper time.

We have found no parameter ranges where the transition rate would be thermal in the sense of the KMS property (7). One situation where approximate thermality might have been expected is near the turning point of a trajectory far from the horizon. However, in this case the transition rate just reduces to that of a geodesic detector in AdS_3, which can be verified not to satisfy the KMS property. These observations are compatible with embedding space arguments which suggest that a detector in AdS_3 should respond thermally only when its proper acceleration exceeds $1/\ell$ [22, 23, 24, 25].

We were however able to analyse the transition rate by a combination of asymptotic methods and numerical methods. Figure 3 shows a plot of the transition rate when the black hole is large and the switch-on and switch-off moments are not close to the white hole and black hole singularities, with the transparent boundary condition at the infinity.

6 Concluding remarks

That the response of a co-rotating detector in the BTZ spacetime is thermal in the co-rotating Hawking temperature was to be expected from the general properties of the Hartle-Hawking vacuum [20, 21]. Our formalism allowed us to analyse this thermal response quantitatively, by a combination of analytic and numerical techniques. We found in particular that the response depends strongly on the choice of the boundary condition at the infinity. We also showed perturbatively that the response loses its thermal character when the detector's angular velocity differs from that of the black hole.

For a detector falling radially into a spinless BTZ hole, we found no parameter space regions where the transition rate would exhibit thermality. The transition rate is again affected by the choice of the boundary condition at the infinity, but this effect appears to be subdominant to those caused by the switching and the motion.

It would be interesting to compare our results for the transition rate in the BTZ spacetime to that in Schwarzschild spacetime. For example, one may expect an inertial detector in Schwarzschild to respond to the Hartle-Hawking vacuum approximately thermally in the asymptotically flat region, owing to the asymptotic flatness of Schwarzschild. We leave these questions subject to future work.

Acknowledgments

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