Black hole formation from a complete regular past for collisionless matter

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We construct initial data for the spherically symmetric Einstein-Vlasov system such that

- in the future a black hole forms
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Dafermos [CMP 2010] has shown that such initial data exist for the Einstein-scalar field system, relying on previous works by Christodoulou. A scalar field is a toy model for matter whereas Vlasov matter is indeed used by astrophysicists.

The Einstein-Vlasov system

This system describes a collisionless ensemble of stars, galaxies or clusters of galaxies, which interact through the gravitational field created collectively.

This system has rich dynamics:

- dispersion for small data
- formation of black holes for large data
- steady states exist (both stable and unstable)
- time periodic oscillations
- exciting dynamics in cosmology

The Einstein-Vlasov system

Let (x^α, p^α) be local coordinates on the tangent bundle of the spacetime (M, g).

The mass shell

$$PM = \{g_{\alpha\beta} p^\alpha p^\beta = -m^2 := -1,\ p^0 > 0\} \subset TM,$$

is invariant under geodesic flow

$$\dot{x}^\alpha = p^\alpha,\ \dot{p}^\alpha = -\Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma.$$

Note that p^0 can be expressed in terms of p^a, $a = 1, 2, 3$ by the mass shell condition.

On PM we thus use coordinates (t, x^a, p^a), $a = 1, 2, 3$.

The Einstein-Vlasov system

The Vlasov equation for $f = f(t, x^a, p^a)$ on PM reads

$$\partial_t f + \frac{p^a}{p^0}\partial_{x^a} f - \frac{1}{p^0}\Gamma^a_{\beta\gamma} p^\beta p^\gamma \partial_{p^a} f = 0.$$

Define the energy momentum tensor by

$$T_{\alpha\beta} := \sqrt{|g|}\int p_\alpha p_\beta f \frac{dp^1 dp^2 dp^3}{-p_0}.$$

The Einstein-Vlasov system reads

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = 8\pi T_{\alpha\beta}.$$

It has nice mathematical properties!

Main results

Theorem

There exists a class of initial data $\mathcal{J}$ for the spherically symmetric Einstein-Vlasov system with the property that black holes form in the future time direction and in the past time direction spacetime is causally geodesically complete.

Corollary 1

A consequence of the proof is the following result which is similar to a result by Christodoulou [CPAM 1991] in the case of a scalar field.

Corollary

Given $\epsilon > 0$, there exists a class $\mathcal{J}_r$ of initial data for the spherically symmetric Einstein-Vlasov system which satisfy

$$\sup_r \frac{\mathring{m}(r)}{r} \leq \epsilon,$$

for which black holes form in the evolution.

Corollary 2

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There exists a class $\mathcal{J}_s$ *of black hole initial data for the spherically symmetric Einstein-Vlasov system such that the corresponding solutions exist for all Schwarzschild time* $t \in (-\infty, \infty)$.

Strategy of proof

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- the data which guarantees the formation of a black hole requires $\frac{2M}{R}$ to be close to 1 and this method only admits data for which $\frac{2M}{R}$ is less than $\sim 1/10$
- the method uses a maximal time gauge and the black hole data is given in Schwarzschild coordinates

Initial data for black hole formation

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To obtain a set of initial data it is sufficient to prescribe an initial density function $\mathring{f} = \mathring{f}(r, w, L),\ L \in [0, L_+]$, such that

$$4\pi^2 \int_0^r \int_{-\infty}^{\infty} \int_0^{L_+} \sqrt{1 + w^2 + \frac{L}{r^2}}\, \mathring{f}(\eta, w, L) dwdLd\eta < \frac{r}{2}.$$

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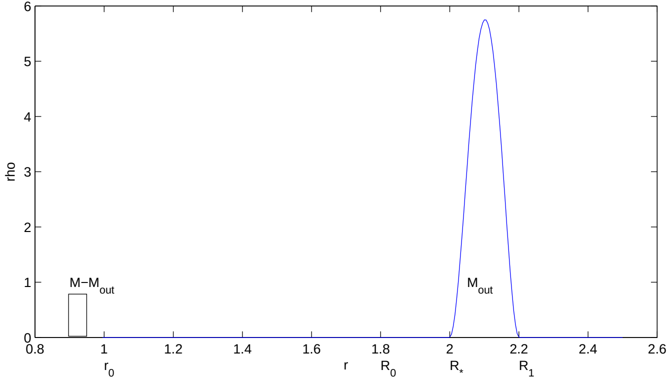
This condition implies that there are no trapped surfaces initially.

Initial data for black hole formation

Let $M > 0$ be given and let $R_0 = 2M$, then there exist numbers

$$0 < r_0 < R_0 < R_* < R_1, \quad L_+ > 0,$$

such that a black hole forms in the evolution if the particles move sufficiently fast inwards in the outer part of the matter.



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Let W_- be large so that the following inequality holds

$$W_- \, e^{\frac{-5M}{2R_*(1-\frac{2M}{R_*})}}(1 - \frac{2M}{R_*})^{3/2} \geq \frac{3}{2} W_*.$$

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Note that $W_- \to \infty$ if $\frac{2M}{R_*} \to 1$.

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By continuity $t_1 > 0$. The main aim is to show that $t_1 = \infty$.

The particles thus continue to go outwards for all times.

The crucial quantity

We define

$$G(s) := \sqrt{1 + W(s)^2 + L/R(s)^2} + W(s),$$

and $\hat{\mu}$ and $\check{\mu}$ by

$$\mu = \int_r^\infty \frac{m(t,\eta)}{\eta^2} e^{2\lambda(t,\eta)} + 4\pi\eta p e^{2\lambda(t,\eta)} d\eta =: \hat{\mu} + \check{\mu}.$$

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The crucial quantity to consider is

$$G(s) e^{\hat{\mu}(s,R(s))} (1 - \frac{2M}{R(s)}).$$

The basic equation

Using the characteristic equations associated to the Vlasov equation we get

$$\begin{aligned}\frac{d}{ds}(Ge^{\hat{\mu}}(1-\frac{2M}{R})) &= -\left[\lambda_t\frac{W}{E}+\mu_r e^{\mu-\lambda}\right]Ge^{\hat{\mu}}(1-\frac{2M}{R}) \\ &+\frac{Le^{\mu-\lambda}}{R^3E}e^{\hat{\mu}}(1-\frac{2M}{R}) \\ &+[\hat{\mu}_r\frac{W}{E}e^{\mu-\lambda}+\hat{\mu}_t]Ge^{\hat{\mu}}(1-\frac{2M}{R}) \\ &+\frac{2M}{R^2}\frac{W}{E}e^{\mu-\lambda}Ge^{\hat{\mu}}.\end{aligned}$$

From this equation it is possible to show that the particles will continue to move outwards for all times.