# How to measure deviation from the Kerr initial data recent progress

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#### The problem

Given a solution to the Einstein vacuum constraint equations,  $(S, h_{ab}, K_{ab})$ , how do we know it is a slice of the Kerr spacetime? If not, can we measure how much it differs?

We will introduce a geometric invariant on the slice, which will measure this deviation from Kerr data.

The invariant is global on the slice, (but local in time).

## Some applications

Expectation: A dynamical black hole settles down to a Kerr/Schwarzschild black hole. To make sense of this one need to measure how close data on a slice is to Kerr data.

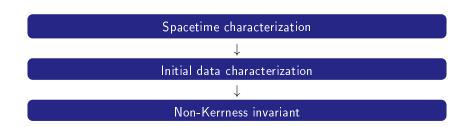
#### Numerical Relativity

Measure how non-Kerrness evolves for numerical spacetimes.

#### Stability of Kerr

A coordinate independent integral over a slice is well suited.

## Spacetime characterization



Use a spacetime characterization based on Killing spinors – here translated to Killing-Yano tensors.

## Spacetime characterization

Let  $(\mathcal{M}, g_{\mu\nu})$  be an orientable and time orientable globally hyperbolic vacuum spacetime.  $\nabla_{\mu}$  denotes the Levi-Civita connection of  $g_{\mu\nu}$ .

#### Killing-Yano tensors

A Killing-Yano tensor  $Y_{\mu\nu}=Y_{[\mu
u]}$  satisfies

$$\nabla_{(\mu} Y_{\nu)\lambda} = 0. \tag{1}$$

Given a Killing-Yano tensor, one automatically gets a Killing vector  $\xi_\mu = \epsilon_\mu{}^{\nu\gamma\lambda} \nabla_\nu Y_{\gamma\lambda}$  of the spacetime.

An integrability condition, will restrict the Weyl tensor.

## Spacetime characterization

#### Theorem

A smooth spacetime  $(\mathcal{M}, g_{\mu\nu})$  is locally isometric to the Kerr spacetime if and only if the following conditions are satisfied:

- (i) there exists a Killing-Yano tensor  $Y_{\mu\nu}$ , with associated Killing vector  $\xi_{\mu}$
- (ii) the spacetime  $(\mathcal{M}, g_{\mu\nu})$  has a stationary asymptotically flat 4-end with non-vanishing mass in which  $\xi_{\mu}$  tends to a time translation.

#### **Theorem**

Let  $(S, h_{ab}, K_{ab})$  be a vacuum initial data set, where S is a Cauchy hypersurface. The development of the initial data set will have a (conformal) Killing-Yano tensor in the domain of dependence of S if and only if

$$\zeta_{ab} \equiv D_{(a}\kappa_{b)} - \frac{1}{3}h_{ab}D_{d}\kappa^{d} + i\epsilon_{(a}{}^{dl}K_{b)d}\kappa_{l} = 0,$$
 (2a)

$$F_{ab} \equiv -C_{(a}{}^{c}\epsilon_{b)c}{}^{d}\kappa_{d} = 0, \tag{2b}$$

are satisfied on S. Here,  $C_{ab} \equiv E_{ab} + iB_{ab}$ .

Furthermore, these conditions gives a complex spacetime Killing vector. Realness of this Killing vector gives a Killing-Yano tensor.

# Approximate Killing-Yano tensors

We want to minimize the  $L^2$  norm of the left hand side of (2a):

$$J = \int_{\mathcal{S}} \zeta_{ab} \bar{\zeta}^{ab} d\mu. \tag{3}$$

The Euler-Lagrange equation reads

$$L(\kappa_a) \equiv D_b \zeta_a{}^b - i\epsilon_{acf} K^{bc} \zeta_b{}^f = 0. \tag{4}$$

A solution,  $\kappa_a$ , to the elliptic equation (4) is called an *approximate* spatial Killing-Yano tensor.

Clearly, any solution to  $\zeta_{ab} = 0$  is also a solution to equation (4).

#### Theorem

Given an asymptotically Schwarzschildean initial data set  $(S, h_{ab}, K_{ab})$ , there exists a smooth unique solution to equation (4) with the same asymptotic behaviour as the solution for Kerr.

Recent work: Also for compact domains.

## The geometric invariant

Let  $\kappa_a$  be a solution to equation (4) as given by the theorem above. With

$$J = \int_{\mathcal{S}} \zeta_{ab} \bar{\zeta}^{ab} \mathrm{d}\mu,\tag{5}$$

$$I_1 \equiv \int_{\mathcal{S}} F_{ab} \bar{F}^{ab} d\mu, \tag{6}$$

the geometric invariant is defined by

$$I \equiv J + I_1. \tag{7}$$

By construction I is coordinate independent.

## Main theorem

#### Theorem

Let  $(S, h_{ab}, K_{ab})$  be a vacuum initial data set with two asymptotically Schwarzschildean ends. Let I be the invariant defined above, where  $\kappa_a$  is the only solution to equation (4) with the same asymptotic behaviour as the solution in the Kerr spacetime. The invariant I vanishes if and only if the development of  $(S, h_{ab}, K_{ab})$  is locally isometric the Kerr spacetime.

## Extra info

 $\kappa_a$  is the pullback of  $-\frac{i}{2}t^{\nu}Y_{\mu\nu} + \frac{1}{4}\epsilon_{\mu\nu\lambda\delta}t^{\nu}Y^{\lambda\delta}$ .

The Killing vector initial data is constructed from  $\kappa_a$  via

$$\zeta = D^a \kappa_a, \quad \zeta_a = \frac{3i}{2} \epsilon_a{}^{bc} D_c \kappa_b + \frac{3}{2} K_{ab} \kappa^b - \frac{3}{2} K^b{}_b \kappa_a. \tag{8}$$

The constraints are

$$R = (K_a{}^a)^2 - K_{ab}K^{ab}, \qquad D^a K_{ab} = D_b K^a{}_a.$$
 (9)

Initial data for the electric and magnetic parts of the Weyl tensor

$$E_{ab} = K_{ca}K^{c}{}_{b} - K_{ab}K^{c}{}_{c} + R_{ab}, \quad B_{ab} = -\epsilon_{b}{}^{dc}D_{c}K_{ad}.$$
 (10)

The space time algebraic condition is

$$0 = -C_{[\mu\nu]}{}^{\delta}{}_{[\lambda}Y_{\rho]\delta} - C_{[\lambda\rho]}{}^{\delta}{}_{[\mu}Y_{\nu]\delta} + C^{\delta}{}_{[\mu\nu][\lambda}Y_{\rho]\delta} + C^{\delta}{}_{[\lambda\rho][\mu}Y_{\nu]\delta} \quad (11)$$