How to measure deviation from the Kerr initial data

recent progress

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The problem

Given a solution to the Einstein vacuum constraint equations, $(\mathcal{S}, h_{ab}, K_{ab})$, how do we know it is a slice of the Kerr spacetime? If not, can we measure how much it differs?

We will introduce a geometric invariant on the slice, which will measure this deviation from Kerr data.

The invariant is global on the slice, (but local in time).

Some applications

Expectation: A dynamical black hole settles down to a Kerr/Schwarzschild black hole. To make sense of this one need to measure how close data on a slice is to Kerr data.

Numerical Relativity

Measure how non-Kerrness evolves for numerical spacetimes.

Stability of Kerr

A coordinate independent integral over a slice is well suited.

Spacetime characterization

Spacetime characterization

↓

Initial data characterization

↓

Non-Kerrness invariant

Use a spacetime characterization based on Killing spinors – here translated to Killing-Yano tensors.

Spacetime characterization

Let $(\mathcal{M}, g_{\mu\nu})$ be an orientable and time orientable globally hyperbolic vacuum spacetime. ∇_μ denotes the Levi-Civita connection of $g_{\mu\nu}$.

Killing-Yano tensors

A Killing-Yano tensor $Y_{\mu\nu} = Y_{[\mu\nu]}$ satisfies

$$\nabla_{(\mu} Y_{\nu)\lambda} = 0. \tag{1}$$

Given a Killing-Yano tensor, one automatically gets a Killing vector $\xi_\mu = \epsilon_\mu{}^{\nu\gamma\lambda} \nabla_\nu Y_{\gamma\lambda}$ of the spacetime.
An integrability condition, will restrict the Weyl tensor.

Spacetime characterization

Theorem

A smooth spacetime $(\mathcal{M}, g_{\mu\nu})$ is locally isometric to the Kerr spacetime if and only if the following conditions are satisfied:

(i) *there exists a Killing-Yano tensor $Y_{\mu\nu}$, with associated Killing vector ξ_μ*

(ii) *the spacetime $(\mathcal{M}, g_{\mu\nu})$ has a stationary asymptotically flat 4-end with non-vanishing mass in which ξ_μ tends to a time translation.*

Theorem

Let $(\mathcal{S}, h_{ab}, K_{ab})$ be a vacuum initial data set, where $\mathcal{S}$ is a Cauchy hypersurface. The development of the initial data set will have a (conformal) Killing-Yano tensor in the domain of dependence of $\mathcal{S}$ if and only if

$$\zeta_{ab} \equiv D_{(a}\kappa_{b)} - \frac{1}{3}h_{ab}D_d\kappa^d + i\epsilon_{(a}{}^{dl}K_{b)d}\kappa_l = 0, \tag{2a}$$

$$F_{ab} \equiv - C_{(a}{}^c\epsilon_{b)c}{}^d\kappa_d = 0, \tag{2b}$$

are satisfied on $\mathcal{S}$. Here, $C_{ab} \equiv E_{ab} + iB_{ab}$.
Furthermore, these conditions gives a complex spacetime Killing vector. Realness of this Killing vector gives a Killing-Yano tensor.

Approximate Killing-Yano tensors

We want to minimize the L^2 norm of the left hand side of (2a):

$$J = \int_{\mathcal{S}} \zeta_{ab}\bar{\zeta}^{ab}\mathrm{d}\mu. \tag{3}$$

The Euler-Lagrange equation reads

$$L(\kappa_a) \equiv D_b\zeta_a{}^b - i\epsilon_{acf}K^{bc}\zeta_b{}^f = 0. \tag{4}$$

A solution, κ_a, to the elliptic equation (4) is called an *approximate spatial Killing-Yano tensor*.

Clearly, any solution to $\zeta_{ab} = 0$ is also a solution to equation (4).

Theorem

Given an asymptotically Schwarzschildean initial data set $(\mathcal{S}, h_{ab}, K_{ab})$, there exists a smooth unique solution to equation (4) with the same asymptotic behaviour as the solution for Kerr.

Recent work: Also for compact domains.

The geometric invariant

Let κ_a be a solution to equation (4) as given by the theorem above. With

$$J = \int_{\mathcal{S}} \zeta_{ab}\bar{\zeta}^{ab}\mathrm{d}\mu, \tag{5}$$

$$I_1 \equiv \int_{\mathcal{S}} F_{ab}\bar{F}^{ab}\mathrm{d}\mu, \tag{6}$$

the geometric invariant is defined by

$$I \equiv J + I_1. \tag{7}$$

By construction I is coordinate independent.

Main theorem

Theorem

Let $(\mathcal{S}, h_{ab}, K_{ab})$ *be a vacuum initial data set with two asymptotically Schwarzschildean ends. Let* I *be the invariant defined above, where* κ_a *is the only solution to equation* (4) *with the same asymptotic behaviour as the solution in the Kerr spacetime. The invariant* I *vanishes if and only if the development of* $(\mathcal{S}, h_{ab}, K_{ab})$ *is locally isometric the Kerr spacetime.*

Extra info

κ_a is the pullback of $-\frac{i}{2}t^\nu Y_{\mu\nu} + \frac{1}{4}\epsilon_{\mu\nu\lambda\delta}t^\nu Y^{\lambda\delta}$.

The Killing vector initial data is constructed from κ_a via

$$\zeta = D^a\kappa_a, \quad \zeta_a = \frac{3i}{2}\epsilon_a{}^{bc}D_c\kappa_b + \frac{3}{2}K_{ab}\kappa^b - \frac{3}{2}K^b{}_b\kappa_a. \tag{8}$$

The constraints are

$$R = (K_a{}^a)^2 - K_{ab}K^{ab}, \qquad D^aK_{ab} = D_bK^a{}_a. \tag{9}$$

Initial data for the electric and magnetic parts of the Weyl tensor

$$E_{ab} = K_{ca}K^c{}_b - K_{ab}K^c{}_c + R_{ab}, \quad B_{ab} = -\epsilon_b{}^{dc}D_cK_{ad}. \tag{10}$$

The space time algebraic condition is

$$0 = \; - C_{[\mu\nu]}{}^\delta{}_{[\lambda}Y_{\rho]\delta} - C_{[\lambda\rho]}{}^\delta{}_{[\mu}Y_{\nu]\delta} + C^\delta{}_{[\mu\nu][\lambda}Y_{\rho]\delta} + C^\delta{}_{[\lambda\rho][\mu}Y_{\nu]\delta} \tag{11}$$