The complete non-spinning effective-one-body metric at linear order in the mass ratio

Enrico Barausse
(University of Guelph)

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based on

E. Barausse, A. Buonanno and A. Le Tiec,
Phys. Rev. D 85, 064010 (2012)

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A. Le Tiec, E. Barausse and A. Buonanno
Phys. Rev. Lett. 108, 131103 (2012)

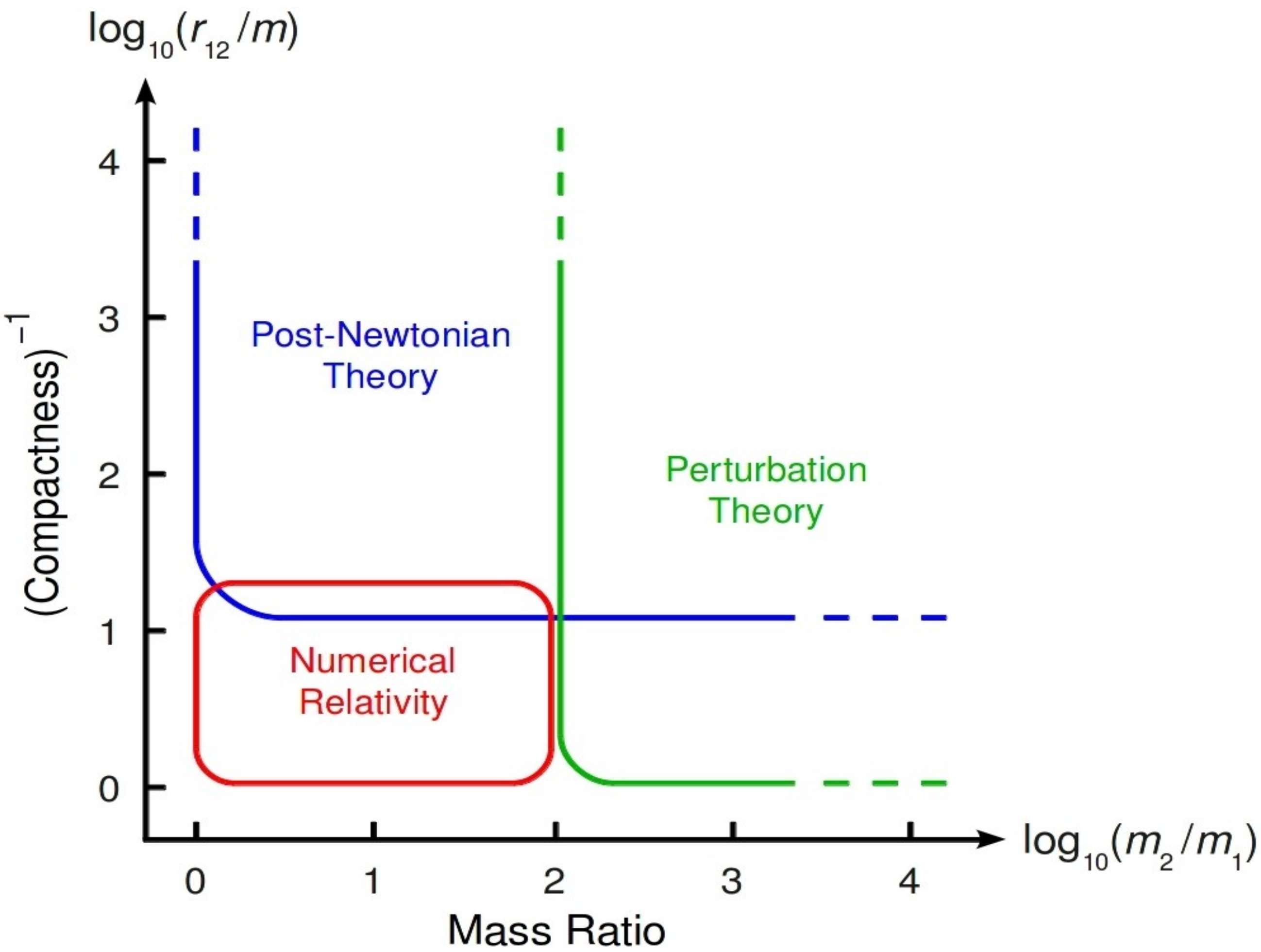
Outline

Follow-up to talk by Alexandre Le Tiec:

- The effective-one-body: a way to patch together the different partial solutions to the two-body problem in GR
- How to use results for the gravitational self-force to determine the effective-one-body metric exactly at linear order in the mass ratio
- Conclusions

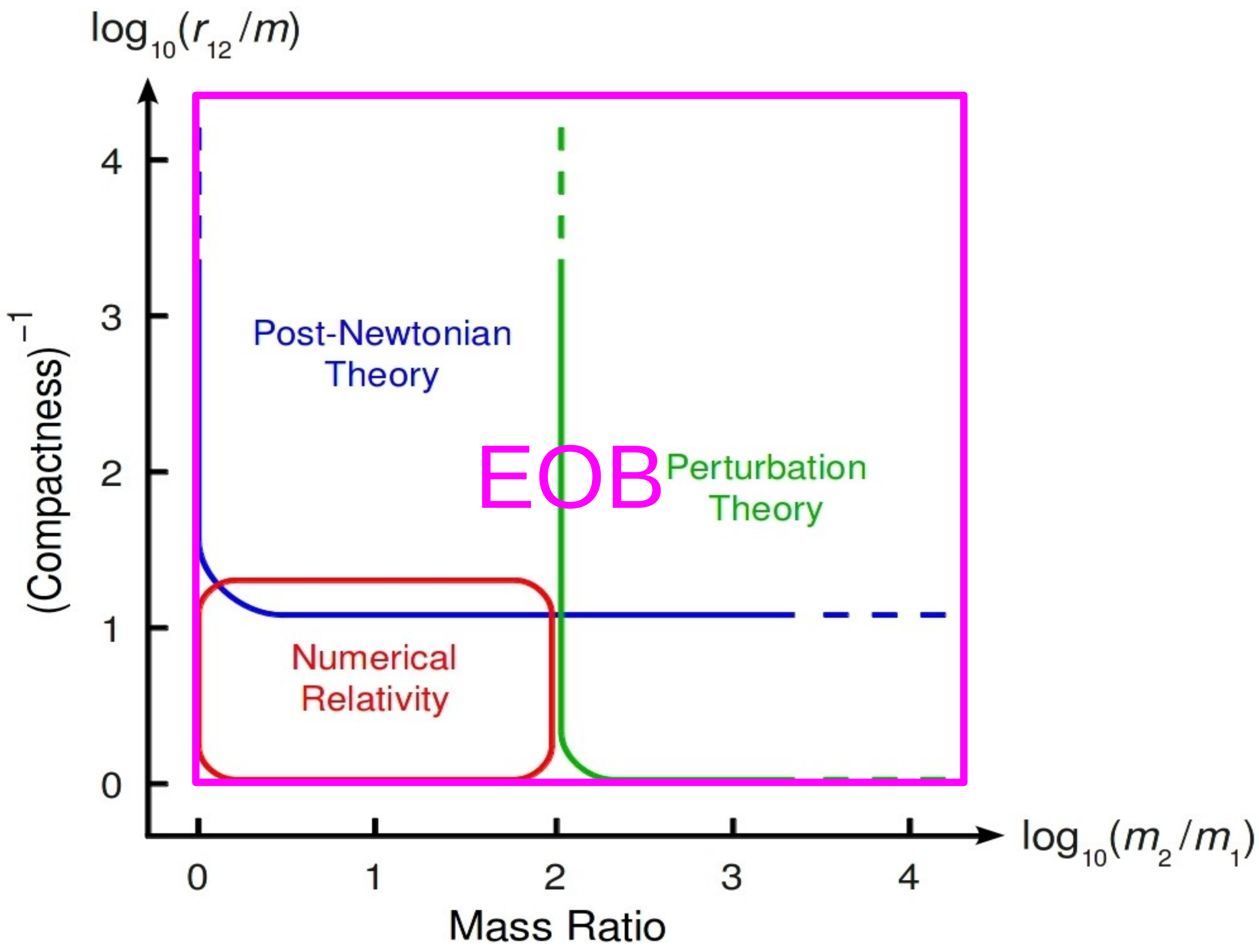
The two-body problem in GR

Techniques have different ranges of validity



The two-body problem in GR

Techniques have different ranges of validity

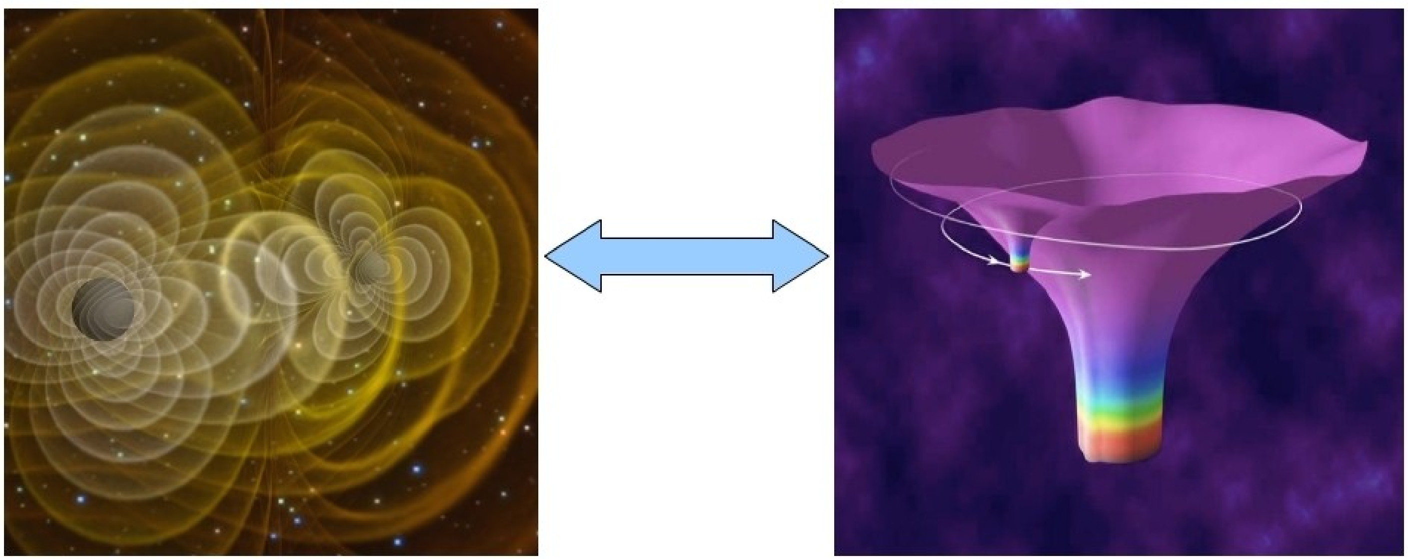


The effective-one-body formalism in a nutshell

- Motivations:

  - understand 2-body problem
  - fast and accurate templates for GW detectors

- Main idea: map 2-body problem into test-particle problem



$m_1 = m_2$ $\qquad$ $m_1 \ll m_2$

Is this mapping possible?

- Newtonian non-spinning binaries can be mapped to non-spinning test-particle with mass $\mu = m_1 m_2/(m_1 + m_2)$ around mass $m = m_1 + m_2$

- Energy levels of positronium ($e^+ - e^-$) can be mapped to those of hydrogen through

$$\frac{E_{\mathrm{H}}}{\mu c^2} = \frac{E_{\mathrm{pos}}^2 - m_1^2 c^4 - m_2^2 c^4}{2 m_1 m_2 c^4}$$

$m_1 = m_2$ is the (anti)-electron's mass

Mapping possible in PN theory for non-spinning BHs (Buonanno & Damour 1999)

- PN Hamiltonian in ADM coordinates ⇒ canonical transformation (does not affect physics) ⇒ "Real'' PN Hamiltonian $H_{PN,real}$

- Particle with mass $\mu = m_1 m_2/(m_1+m_2)$ around a $m=m_1+m_2$ *deformed* Schwarzschild BH ("effective problem'') has Hamiltonian H_{eff}, which satisfies

$$\frac{H_{\rm eff}}{\mu c^2} = \frac{H^2_{\rm PN,\,real} - m_1^2 c^4 - m_2^2 c^4}{2 m_1 m_2 c^4} \quad \text{(up to 3 PN)} \ (*)$$

- H_{eff} can be calculated at all PN orders (deformed Schwarzschild metric given at all PN orders) ⇒ invert Eq (*) and get "real" Hamiltonian valid at all PN orders:

$$H_{\rm real} = m\sqrt{1 + 2\frac{\mu}{m}\left(\frac{H_{\rm eff}}{\mu} - 1\right)}\,.$$

The EOB Hamiltonian for non-spinning BHs

$$H_{\rm real} = M\sqrt{1 + 2\nu\left(\frac{H_{\rm eff}}{\mu} - 1\right)}$$

$$H^{\rm eff}(r, p_r, p_\phi) \equiv \mu\,\widehat{H}^{\rm eff} = \mu\sqrt{A(r)\left[1 + \frac{A(r)}{D(r)}p_r^2 + \frac{p_\phi^2}{r^2} + 2(4-3\nu)\,\nu\,\frac{p_r^4}{r^2}\right]}$$

$$\mathrm{d}s_{\rm eff}^2 = -A(r)\,\mathrm{d}t^2 + B(r)\,\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$$

$$A(u) = 1 - 2u + 2\nu\,u^3 + \left(\frac{94}{3} - \frac{41}{32}\pi^2\right)\nu\,u^4 + \mathcal{O}(u^5)$$

$$\bar{D}(u) = 1 + 6\nu\,u^2 + (52 - 6\nu)\,\nu\,u^3 + \mathcal{O}(u^4) \qquad \bar{D} \equiv (AB)^{-1} \qquad u \equiv M/r$$

Potentials can be rewritten in different ways at higher PN orders (e.g. Pade resummation, “log” resummation)

More on the EOB

- Mapping also possible for systems of *spinning* BHs in PN theory (Damour 2001, Damour et al 2008, Barausse and Buonanno 2010, 2011, Nagar 2011)

- Can be supplemented with model for waveforms (and thus fluxes) and recipe to attach QNMs → inspiral-merger-ringdown waveforms for spinning BHs implemented in LIGO and used for searches (Taracchini et al 2012)

- EOB has free parameters: fixed by comparison to NR waveforms and SF results (PN theory already included in the mapping and waveforms)

The EOB calibration with SF results

- Calculation of the SF-induced ISCO shift for a particle in Schwarzschild (Barack & Sago 2009)

$$m\Omega_{\rm ISCO} = 6^{-3/2}\left[1+\nu C_\Omega + \mathscr{O}(\nu^2)\right] \qquad C_\Omega = 1.2512(4)$$

$$m = m_1 + m_2, \qquad \nu = \frac{m_1 m_2}{m^2} = \frac{m_1}{m_2} + O\left(\frac{m_1}{m_2}\right)^2,$$

  Used to calibrate free parameters regulating high (unknown) PN orders in A(u) [Damour 2010, Barausse & Buonanno 2010]

- Effect of SF on periastron precession (Barack and Sago 2009)

  Used to constrain combination of A(u) and D(u) [Barack, Damour & Sago 2010]

Another result: the redshift (Detweiler 2008)

- Spacetime of binary that remains on circular orbit forever has helical Killing vector (reasonable approx to real binaries in adiabatic regime)

$$K = \partial_t + \Omega \partial_\phi$$

- Effect of the SF on the projection of the 4-velocity of a circular orbit in Schwarzschild on helical killing vector calculated with high accuracy (Detweiler 2008)

$$z \equiv -u \cdot K = \sqrt{1-3x} + \nu z_{SF}(x) + O(\nu)^2$$

$$x = (m\Omega)^{2/3}$$

The binding energy and angular momentum of circular non-spinning binaries (Le Tiec, Barausse, Buonanno 2012)

$$\hat{E} = \left(\frac{1-2x}{\sqrt{1-3x}} - 1\right) + \nu\, E_{\mathrm{SF}}(x) + \mathscr{O}(\nu^2)\,,$$

$$\hat{J} = \frac{1}{\sqrt{x(1-3x)}} + \nu\, J_{\mathrm{SF}}(x) + \mathscr{O}(\nu^2)\,,$$

$$\hat{E} \equiv (E - m)/\mu$$

$$\hat{J} \equiv J/(m\mu)$$

$$\nu = \mu/m = \frac{m_1 m_2}{(m_1+m_2)^2}$$

$$E_{\mathrm{SF}}(x) = \frac{1}{2}\, z_{\mathrm{SF}}(x) - \frac{x}{3}\, z'_{\mathrm{SF}}(x) - 1 + \sqrt{1-3x} + \frac{x}{6}\frac{7-24x}{(1-3x)^{3/2}}\,,$$

$$J_{\mathrm{SF}}(x) = -\frac{1}{3\sqrt{x}}\, z'_{\mathrm{SF}}(x) + \frac{1}{6\sqrt{x}}\frac{4-15x}{(1-3x)^{3/2}}\,.$$

- $z_{SF}(x)$ calculated numerically for $x \le 1/5$ ($r \gtrsim 5m$) with accuracy 10^{-6} (Detweiler 2008)
- Data fitted to within 10^{-5} by

$$z_{\mathrm{SF}}(x) = 2x\,(1 + a_1 x + a_2 x^2)/(1 + a_3 x + a_4 x^2 + a_5 x^3)$$

$a_1 = -2.18522$, $a_2 = 1.05185$, $a_3 = -2.43395$, $a_4 = 0.400665$, and $a_5 = -5.9991$

Including the info from the binding energy in the EOB

- Write EOB potentials as $$A(u) = 1 - 2u + \nu\, A_{\rm SF}(u) + \mathcal{O}(\nu^2)\,,$$ $$\bar{D}(u) = 1 + \nu\, \bar{D}_{\rm SF}(u) + \mathcal{O}(\nu^2)\,.$$

- Calculate EOB energy (at next-to-leading order in mass ratio) and compare with expression coming from redshift

$$\hat{E}_{SF}(x) = \hat{E}_{EOB}(x) \quad \Rightarrow$$

$$2x\, A'_{\rm SF}(x) - 3\frac{1-4x}{1-3x} A_{\rm SF}(x) = x\,\frac{1-6x}{1-3x} + \sqrt{1-3x}\, \times$$
$$\left[2x\, z'_{\rm SF}(x) - 3 z_{\rm SF}(x) + x\,\frac{1-5x+12x^2}{(1-3x)^2}\right]$$

Including the info from the binding energy in the EOB

$$A_{\rm SF}(x) = \sqrt{1-3x}\; z_{\rm SF}(x) - x\left(1 + \frac{1-4x}{\sqrt{1-3x}}\right)$$

- Because $\frac{\partial E_{EOB}}{\partial J_{EOB}} = \Omega = \frac{\partial E_{SF}}{\partial J_{SF}}$, sufficient to ensure $\hat{J}_{EOB}(x) = \hat{J}_{SF}(x)$
- Automatically reproduces SF ISCO shift (because ISCO= minimum of the energy)
- Completely determines A(u) and EOB dynamics for circular orbits at linear order in mass ratio!

The radial potential D

- SF effect on periastron precession [Barack, Damour & Sago 2010]

$$\Omega_r \equiv \frac{2\pi}{P} \qquad \Omega_\phi \equiv \frac{1}{P}\int_0^P \dot{\phi}(t)\,\mathrm{d}t = K\,\Omega_r \qquad \Delta\Phi/(2\pi) = K - 1$$

$$W \equiv 1/K^2 = 1 - 6x + \nu\,\rho_{\mathrm{SF}}(x) + \mathcal{O}(\nu^2)$$

$$\bar{D}_{\mathrm{SF}}(x) = \frac{1}{1-6x}\left[\rho_{\mathrm{SF}}(x) + 4x\left(\frac{1-2x}{\sqrt{1-3x}} - 1\right) - A_{\mathrm{SF}}(x) - x\,A'_{\mathrm{SF}}(x) - \frac{x}{2}(1-2x)\,A''_{\mathrm{SF}}(x)\right]$$

- $\rho_{SF}(x)$ known as fit
- Completes knowledge of EOB metric and quasi-circular dynamics at linear order in mass-ratio

Check converge of PN orders

- EOB potentials A(u) and D(u) now known at all PN orders (at linear order in mass-ratio)
- We can truncate them at nPN order and compare performance of EOB vs PN theory

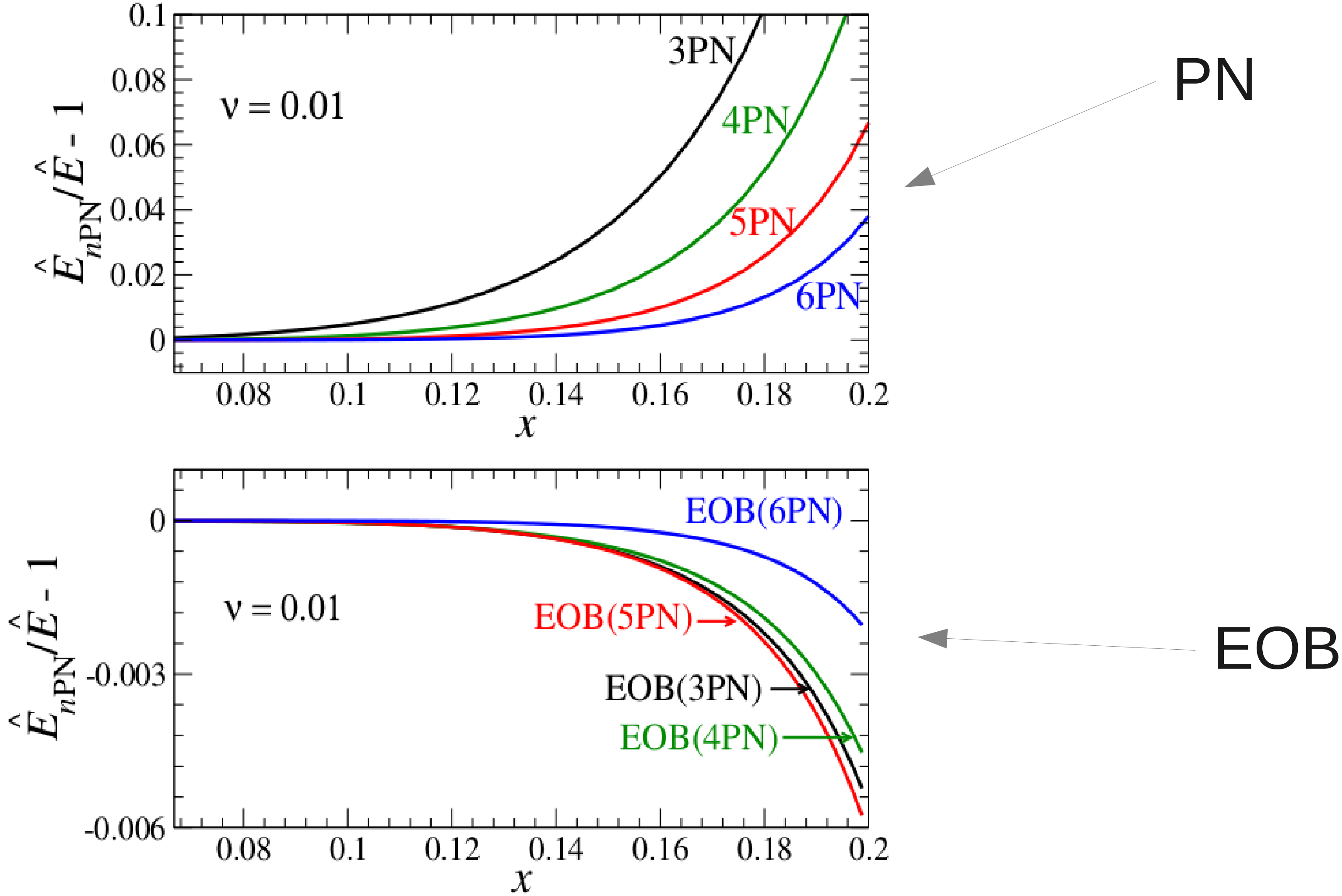
$$A(u) = 1 - 2u + 2\nu\, u^3 + \left(\frac{94}{3} - \frac{41\pi^2}{32}\right)\nu\, u^4$$
$$+ \nu \left[a_5(\nu) + a_5^{\ln}(\nu)\,\ln u\right] u^5$$
$$+ \nu \left[a_6(\nu) + a_6^{\ln}(\nu)\,\ln u\right] u^6$$
$$+ \nu \left[a_7(\nu) + a_7^{\ln}(\nu)\,\ln u\right] u^7 + o(u^7)$$

$$a_5(0) = +23.50190(5)\,,$$
$$a_6(0) = -131.72(1)\,,$$
$$a_7(0) = +118(2)\,,$$
$$a_7^{\ln}(0) = -255.0(5)\,.$$

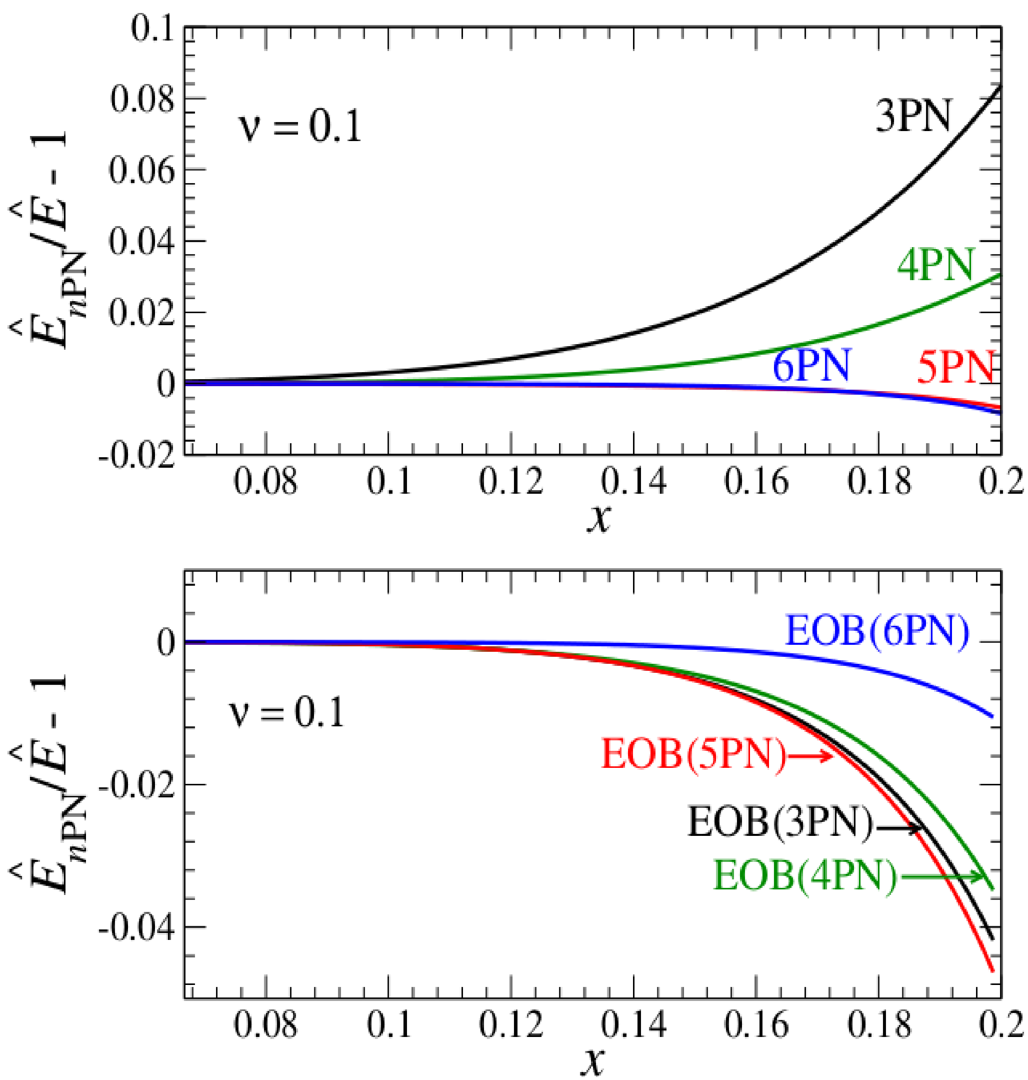
$$\bar{D}(u) = 1 + 6\nu\, u^2 + (52 - 6\nu)\,\nu\, u^3$$
$$+ \nu \left[\bar{d}_4(\nu) + \bar{d}_4^{\ln}(\nu)\,\ln u\right] u^4$$
$$+ \nu \left[\bar{d}_5(\nu) + \bar{d}_5^{\ln}(\nu)\,\ln u\right] u^5 + o(u^5)$$

$$\bar{d}_4(0) = +226.0^{+7}_{-4}\,,$$
$$\bar{d}_4^{\ln}(0) = +\frac{592}{15}\,,$$
$$\bar{d}_5(0) = -649^{-1200}_{+400}\,,$$
$$\bar{d}_5^{\ln}(0) = -\frac{1420}{7}\,.$$

EOB and PN vs approximants for the binding energy



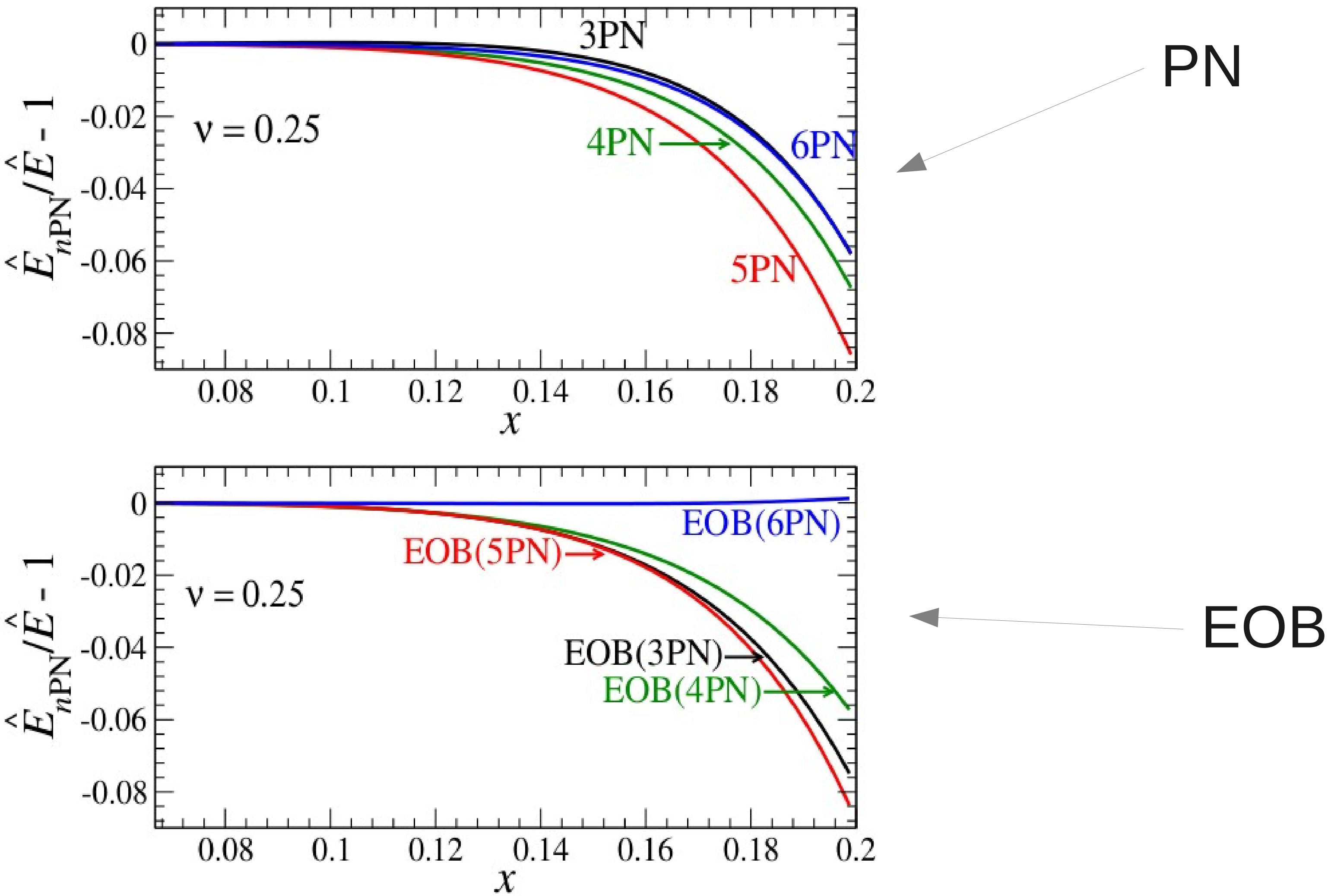
EOB and PN vs approximants for the binding energy



PN

EOB

EOB and PN vs approximants for the binding energy



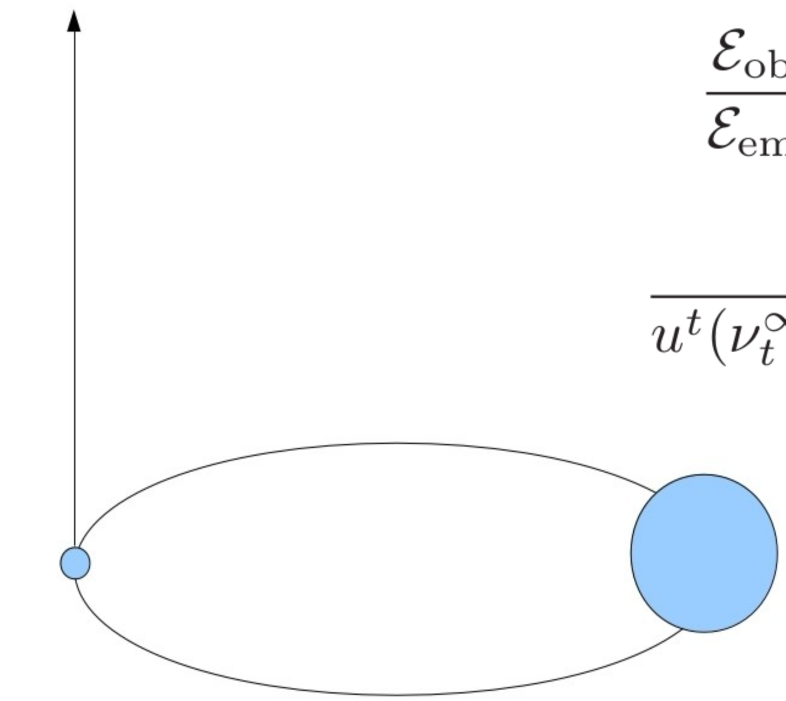
Conclusions

- We determined the EOB metric exactly at linear order in mass ratio, for non-spinning BH binaries
- This metric completely determines dynamics of quasi-circular binaries in the EOB model!
- With this metric, the EOB reproduces all known SF results:
  - Binding energy and angular momentum (Le Tiec, Barausse, Buonanno 2012)
  - Periastron precession (Barack, Damour & Sago 2010)
  - ISCO shift (Barack & Sago 2009)

THE END

Another result: the redshift (Detweiler 2008)

Can be interpreted as redshift of photon emitted by particle along z axis → "redshift" observable



$$\frac{\mathcal{E}_{\rm ob}}{\mathcal{E}_{\rm em}} = \frac{u^a_{\rm ob}\nu_a}{u^a_{\rm em}\nu_a} = \frac{u^t_\infty \nu^\infty_t}{u^t(k^a\nu_a)_{\rm em}} = \frac{\nu^\infty_t}{u^t(k^a\nu_a)^\infty}$$

$$\frac{\nu^\infty_t}{u^t(\nu^\infty_t + \Omega\nu^\infty_\phi)} = \frac{1}{u^t} - \frac{\Omega\nu^\infty_\phi}{u^t(\nu^\infty_t + \Omega\nu^\infty_\phi)} = \frac{1}{u^t}$$

$$u = u^t\partial_t + u^\phi\partial_\phi = u^t K$$

$$-1 = u\cdot u = (u^t)^2 K\cdot K$$

$$z = -u\cdot K = 1/u^t$$