Dirac equation in curved spacetime and hidden symmetries

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Introduction

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- Application to the Dirac equation in curved space-time using Conformal Killing Yano tensors
- Recent notable example: full separation of variables in higher dimensional rotating black hole metrics

Plan of the talk

1. Hidden Symmetries and the Dirac Equation

2. Rotating Black Holes in Higher Dimension

3. Future perspectives and conclusions

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CCKY tensors form an algebra under wedge product [Krtouš, Kubizňák, Page, Frolov 2007]

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$$S_{\mathrm{e}} = \sum_{p\ \mathrm{odd}} \frac{1}{(p-1)!} \Big[\gamma^{a_1\ldots a_{p-1}} (f_p)^b{}_{a_1\ldots a_{p-1}} \nabla_b + \frac{1}{2(p+1)^2} \gamma^{a_1\ldots a_{p+1}} (df_p)_{a_1\ldots a_{p+1}} \Big] \,,$$

$$S_{\mathrm{o}} = \sum_{p\ \mathrm{even}} \frac{1}{p\,!} \Big[\gamma^{ba_1\ldots a_p} (h_p)_{a_1\ldots a_p} \nabla_b - \frac{p(n-p)}{2(n-p+1)} \gamma^{a_1\ldots a_{p-1}} (\delta h_p)_{a_1\ldots a_{p-1}} \Big] \,,$$

[Benn & Charlton 1997, Benn & Kress 2004],
[Cariglia, Krtouš, Kubizňák 2011]

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- Killing coordinates ψ_i, $i = 0, \ldots, N - 1 + \epsilon$ + other coordinates x_μ, $\mu = 1, \ldots, N$

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- Dirac: theory of separability not well established. However explicit separation of variables in Kerr-NUT-(A)dS achieved in [Oota, Yasui 2008].
- Show separation for Dirac in Kerr-NUT-(A)dS is explained by a complete set of mutually commuting operators

Intrinsic Separability of Dirac Equation 1

Operators $K_k = K_{\xi^{(k)}}$ and $M_j = M_{\frac{1}{j!}h^{(j)}}$ include the Dirac operator $D = M_0$ and mutually commute [Cariglia, Krtouš, Kubizňák 2011] . They can be simultaneously diagonalised

$$K_k \xi = i \, \Psi_k \xi \ , \quad M_j \xi = \mathcal{X}_j \xi \ ,$$

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Dirac equation reduces to separate equations for each χ_ν spinor:

$$\left[\left(\frac{d}{dx_\nu} + \frac{X'_\nu}{4X_\nu} + \frac{\tilde{\Psi}_\nu}{X_\nu}\iota_{\langle\nu\rangle} + \frac{\epsilon}{2x_\nu}\right)\sigma_{\langle\nu\rangle} - \frac{\left(-\iota_{\langle\nu\rangle}\right)^{N-\nu}}{\sqrt{|X_\nu|}}\left(\epsilon\frac{i\sqrt{-c}}{2x_\nu^2} + \tilde{\mathcal{X}}_\nu\right)\right]\chi_\nu = 0\,.$$

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Same as solution found in [Oota, Yasui 2008]

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  can be found in standard separated form (no R factor)
- operators $\tilde{M}_j$ are operators M_j in the 'R-representation'

  $$\tilde{\gamma}^a = R^{-1} \gamma^a R \,.$$

[Cariglia, Krtouš, Kubizňák 2011.2]

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- Much more to be done!

Thank you!

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where α is an arbitrary inhomogeneous form, and S_ω is defined in terms of an inhomogeneous CKY form ω as

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If $D\psi = 0 \rightarrow D(S_\omega\psi) = 0$ [Benn & Charlton 1997, Benn & Kress 2004]