Dirac equation in curved spacetime and hidden symmetries

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 ⇔ hidden symmetries of dynamics

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- Recent notable example: full separation of variables in higher dimensional rotating black hole metrics

Plan of the talk

Hidden Symmetries and the Dirac Equation

Rotating Black Holes in Higher Dimension

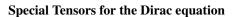
3 Future perspectives and conclusions

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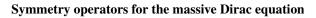
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CCKY tensors form an algebra under wedge product [Krtouš, Kubizňák, Page, Frolov 2007]



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$$\begin{split} S_{\rm e} &= \sum_{p \text{ odd}} \frac{1}{(p-1)!} \Big[\gamma^{a_1 \dots a_{p-1}} (f_p)^b_{a_1 \dots a_{p-1}} \nabla_b + \frac{1}{2(p+1)^2} \gamma^{a_1 \dots a_{p+1}} (df_p)_{a_1 \dots a_{p+1}} \Big] , \\ S_{\rm o} &= \sum_{p \text{ odd}} \frac{1}{p!} \left[\gamma^{ba_1 \dots a_p} (h_p)_{a_1 \dots a_p} \nabla_b - \frac{p(n-p)}{2(n-p+1)} \gamma^{a_1 \dots a_{p-1}} (\delta h_p)_{a_1 \dots a_{p-1}} \right] , \end{split}$$

[Benn & Charlton 1997, Benn & Kress 2004], [Cariglia, Krtouš, Kubizňák 2011]

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- Killing coordinates ψ_i , $i = 0, \dots, N-1+\epsilon$ + other coordinates x_μ , $\mu = 1, \dots, N$

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- Dirac: theory of separability not well established. However explicit separation of variables in Kerr-NUT-(A)dS achieved in [Oota, Yasui 2008].
- Show separation for Dirac in Kerr-NUT-(A)dS is explained by a complete set of mutually commuting operators

Operators $K_k=K_{\xi^{(k)}}$ and $M_j=M_{\frac{1}{J!}h^{(j)}}$ include the Dirac operator $D=M_0$ and mutually commute [Cariglia, Krtouš, Kubizňák 2011] . They can be simultaneously diagonalised

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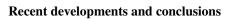
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- Much more to be done!

Thank you!

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If $D\psi = 0 \rightarrow D(S_{\omega}\psi) = 0$ [Benn & Charlton 1997, Benn & Kress 2004]