Geometrostatics: The geometry of static spacetimes

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100 years after Einstein Prague
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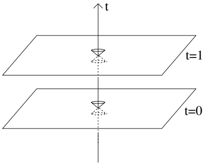
Aim

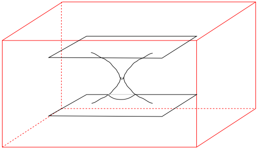
Gain better understanding of static isolated relativistic systems:

- their geometry
- their asymptotics
- the behavior of test particles
- their total mass and center of mass
- their Newtonian limit

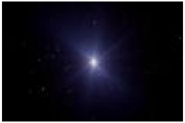
Static Isolated Gravitating Systems

Geometrostatics: static isolated gravitating systems with finitely extended sources





They model individual stars, black holes, and static configurations[1]:



[1] sources: portal-der-erinnerung.de, kath-blog.de

Lapse and 3-Metric

Generic static spacetimes can be canonically decomposed into $M^4 = \mathbb{R} \times M^3$,

$$^4g = -N^2c^2dt^2 + {}^3g,$$

with induced Riemannian metric ^3g on M^3 and

$$N := \sqrt{-^4g(X,X)} > 0$$

the lapse function of the static spacetime.

Here, X is the timelike hypersurface-orthogonal Killing vector.

PDEs of Geometrostatics (Vacuum Region)

The symmetry reduced (vacuum) Einstein equations on M^3 read

$$\begin{aligned} N\,{}^3\mathrm{Ric} &= {}^3\nabla^2 N \\ {}^3\triangle N &= 0. \end{aligned}$$

They are called the static metric equations.

Variables:
N lapse function (length of Killing vector),
${}^3g$ induced Riemannian 3-metric,
${}^3\mathrm{Ric}$ induced Ricci curvature,
${}^3\nabla^2$ induced covariant Hessian,
${}^3\triangle$ induced covariant Laplace-Beltrami operator.

Physical Interpretation

- $(M^3, {}^3g)$ is a time slice, i.e. the state of a static system at any point of time in the eyes of the chosen observer X
- N describes how to measure time in order to “see” staticity.

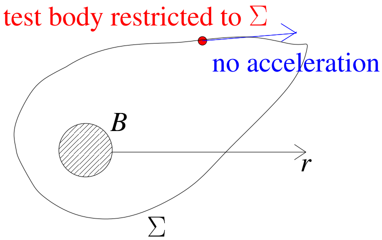
We claim:

  - The level sets of N relate to the dynamics of test particles in the spacetime.
  - N is unique (for a given static 3-metric ${}^3g$).

Equipotential Surfaces in Newtonian Gravity

A level set of the Newtonian potential U is called a equipotential surface in Newtonian Gravity.

Equipotential surfaces are the only surfaces $\Sigma \subseteq \mathbb{R}^3$ such that any test body constrained to Σ and not subject to any exterior forces is not accelerated.



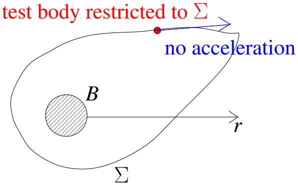
Equipotential Surfaces in Geometrostatics

Definition

A surface $\Sigma \subset M^3$ is called an equipotential surface in $(M^3, {}^3g, N)$ if every constrained test particle (timelike 4-geodesic) of the form

$$\mu(\tau) = (t(\tau), x(\tau))$$

with $x(\tau) \in \Sigma$ inside the spacetime $(\mathbb{R} \times M^3, {}^4g = -N^2c^2dt^2 + {}^3g)$ is a geodesic in Σ with respect to the induced 2-metric.



The Role of the Level Sets of N

Theorem (C.)

A surface $\Sigma \subset M^3$ is an equipotential surface in $(M^3, {}^3g, N)$ if and only if $N \equiv$ const on Σ, i.e. iff Σ is a level set of N.

Proof uses that the relevant curves are critical points of

$$L(\mu) := \int_a^b [|\dot{\mu}(\tau)| + \sigma(V \circ x(\tau) - V_0)] \, d\tau$$

where V is any function on a neighborhood of Σ with $dV \neq 0$ and $V \equiv V_0$ on Σ and σ is a Lagrange multiplier.

Uniqueness of N in Geometrostatics

Theorem (C.)

The equipotential surfaces (and the Lorentzian metric) of a static gravitational system are in fact independent of the lapse function N. In other words, if $(\mathbb{R} \times M^3, {}^4g)$ is a static solution of Einstein's equation (in vacuum), then it is uniquely characterized by its induced 3-metric ${}^3g$.

Proof: The constraint equations reduce to ${}^3\mathrm{R} = 0$ in our case ($K \equiv 0$), so the claim for ${}^4g$ follows Choquet-Bruhat's famous theorem on the IVP for the Einstein equations. Equipotential surfaces only depend on ${}^4g$. □

Uniqueness of N ctd.

Theorem (C.)

Let $({}^3g, N)$ and $({}^3g, \tilde{N})$ solve the static metric equations with $N, \tilde{N} \to 1$ as $r \to \infty$ and suppose that ${}^3g$ is asymptotically flat but not entirely flat. Then $N \equiv \tilde{N}$.

Proof: (vacuum part here, to combine with elliptic theory)

- Levels of $N, \tilde{N}$ are each the only equipotential surfaces $\Rightarrow \tilde{N} = f \circ N$ for some function $f : \mathbb{R} \to \mathbb{R}$.
- $0 = {}^3\triangle \tilde{N} = {}^3\triangle(f \circ N) = f'' \circ N \, \|{}^3\mathrm{grad} N\|^2_{{}^3g} + f' \circ N \underbrace{{}^3\triangle N}_{=0}$ so that $f'' = 0$ and thus $\tilde{N} = \alpha N + \beta$ with $\alpha, \beta \in \mathbb{R}$.
- ${}^3\nabla^2 \tilde{N} = \tilde{N}\, {}^3\mathrm{Ric} = (\alpha N + \beta)\, {}^3\mathrm{Ric} = \alpha\, {}^3\nabla^2 N + \beta\, {}^3\mathrm{Ric} = {}^3\nabla^2 \tilde{N} + \beta\, {}^3\mathrm{Ric}$ and ${}^3\mathrm{Ric} \neq 0$ gives $\beta = 0$.
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$\square$

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□

Conclusion

- The lapse function N of a geometrostatic system is unique.
- Its level sets behave like equipotential surfaces in Newtonian Gravity.
- N can be used to define force on test particles.

Thank you for your attention!