Loop Quantum Cosmology:

Anisotropy and singularity resolution

Alejandro Corichi

Centro de Ciencias Matematicas, UNAM

Einstein Centennial Conference, Prague, June 2012

In collaboration con A. Karami and E. Montoya

MOTIVATION

- **General Relativity predicts singularities**
- **Is there a quantum theory that avoids the Big Bang?**
- **But, what is a singularity?**
- **Standard Quantum Cosmology (WDW) does not solve it.**
- **Isotropic Loop Quantum Cosmology very successful**
- **What can we learn from anisotropic models?**

1. What to Expect?

From a successful quantum theory of gravity we demand singularities to be resolved.

But, what does that mean?
It means that we need an operational notion of when a singularity is avoided in a quantum theory. If we have a quantum formalism,

does it pass this test?

What is a singularity in GR and how do we know where a singularity appears?
How generic are they?

Intuitively a singularity is where bad things happen, and that is normally signaled by the curvature blowing up. The only meaningful quantities in this respect are curvature scalars,

$$R, \qquad R_{ab}R^{ab}, \qquad R_{abcd}R^{abcd}, \ \ldots$$

If one of this curvature scalars divergence at a point, then this is a signal that a singularity might be present.

But the precise definition of when a singularity exists is more subtle. A spacetime is said to be singular if it is *not* geodesically complete.

The singularity theorems of Hawking, Penrose and Geroch tell us that, under reasonable assumptions, singularities are generic.

They occur in the interior of collapsing objects (black holes) and in the past of expanding cosmological solutions, in the *Big Bang*. Key is the Raychaudhuri Equation:

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = -\frac{1}{3}\theta^2 - \sigma^2 - R_{ab}v^a v^b$$

where θ is the expansion of a congruence of timelike geodesics v^a (hypersurface orthogonal), and $\sigma^2 = \sigma_{ab}\sigma^{ab}$ is the shear.

Note that, if $R_{ab}v^a v^b \geq 0$, then the RHS is always negative. If $\theta < 0$ at some time, then, in a finite proper time it will blow up ($\theta \to -\infty$). If matter satisfies certain energy conditions, then $R_{ab}v^a v^b \geq 0$ and the contracting congruences will have a conjugate point in the future.

The expectation is that, by quantizing the gravitational field, one will then have:
i) A complete theory unifying gravity and the quantum.
ii) The singularities resolved.

First step. $(M, g) \longrightarrow (M = \Sigma \times \mathbb{R}; (q_{ab}, K_{ab}))$
The theory is 'constrained'. This means that there are constraints relating the canonical variables: $C_i(q, \pi) = 0$,
that the initial data must satisfy.

One has to promote (1st class) constraints to operators:

$$\hat{C}_i \cdot \Psi(q_{ab}) = 0$$

Solutions to the constraint equations are physical states on $\mathcal{H}_{\rm phys}$.

Then, one needs to define an inner product on the space of solutions $\mathcal{H}_{\rm phys}$.

Finally, one needs to identify physical observables.

Look at geometrical quantities θ and σ^2 (and also matter density ρ and curvature scalars) to test singularity resolution.

Define physical operators and check their properties.

Are they bounded?

Simplest model: Homogeneous and Isotropic FRW cosmology given by:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\mathrm{d}\Sigma^2$$

If matter is 'decent', the function $a(t) \to 0$ when $t \to 0$, which is the singularity.

Simplest matter content: massless scalar field ϕ, satisfying the Klein-Gordon equation. Phase space:

Gravitational degrees of freedom: (a, p_a)

Matter degrees of freedom: (ϕ, p_ϕ)

Loop quantization (as in LQG) has provided a model that can be exactly solved and for which:

- All states undergo a bounce: $\langle V \rangle_\phi \approx V_o \cosh(\phi - \phi_o)$.
- Matter density operator has an absolute upper bound: $|\hat{\rho}| < \rho_{\rm crit} \approx .41\rho_{\rm pl}$.
- Dynamics of semiclassical states are well captured by effective theory:
  $H^2 = \frac{8\pi G}{3}\,\rho\,(1 - \rho/\rho_{\rm crit})$
  Dynamics is generated by an *effective* Hamiltonian $\mathcal{H}_{\rm eff}$.
- GR dynamics is recovered as we go away from the Planck scale $\rho < \frac{\rho_{\rm crit}}{10}$.
- Expansion θ is also absolutely bounded

2. Anisotropies

Loop quantum cosmology has been extended to the simplest anisotropic cosmological models: Bianchi I and II (Ashtekar, Wilson-Ewing) and Bianchi IX (Wilson-Ewing, AC & Karami).

New issues to consider:

- Is the bounce generic?

- We have now anisotropy/Weyl curvature, how does it behave near the singularity/bounce?

- Can we have different kind of bounce, say dominated by shear?

- Are geometric scalars θ, σ and density ρ absolutely bounded?

Bianchi I in GR.

1. Classically the Bianchi I case with massless scalar field has the line element, $ds^2 = -dt^2 + t^{2k_1}dx^2 + t^{2k_2}dy^2 + t^{2k_3}dz^2$, where k_i are the Kasner exponents satisfying the constraints $k_1 + k_2 + k_3 = 1$ and $k_1^2 + k_2^2 + k_3^2 + k_\phi^2 = 1$. With $k_i := H_i/|\theta|$
2. $\Omega := \frac{24\pi G\rho}{\theta^2}$, and $\Sigma^2 := \frac{3\sigma^2}{2\theta^2}$ are conserved and satisfy: $\Omega + \Sigma^2 = 1$.
3. The isotropic solution is contained in Bianchi I.

Bianchi II in GR.

$$ds^2 = -N(\tau)^2 d\tau^2 + a_1(\tau)^2 (dx - \alpha z dy)^2 + a_2(\tau)^2 dy^2 + a_3(\tau)^2 dz^2 \,,$$

1. Ω, Σ^2 and K are conserved and satisfy $\Omega + \Sigma^2 + K = 1$. With $K = \frac{3x^2}{4\theta^2}$
2. Bianchi II is past and future asymptotic to Bianchi I (Jacobs solution).
3. The Bianchi I model (and the isotropic solution) is not contained within Bianchi II.

Effective Equations. Some results in Bianchi I.

1. The effective dynamics for the flat isotropic universe is included into the Bianchi I dynamics.
2. In the isotropic case all the solutions to the effective equations have a maximal density equal to the critical density $\rho_{\rm crit} = \frac{3}{8\pi G\gamma^2\lambda^2} \approx 0.41\rho_{\rm pl}$.
3. When the shear is different from zero, Bianchi I solutions have maximal density less than the critical density.
4. Classical and effective solutions are equal far away from the bounce (not so far). The effective solutions to Bianchi I connect two classical solutions with Kasner exponents related by $k_1, k_2, k_3 \to k_1 - \frac{2}{3}, k_2 - \frac{2}{3}, k_3 - \frac{2}{3}$.

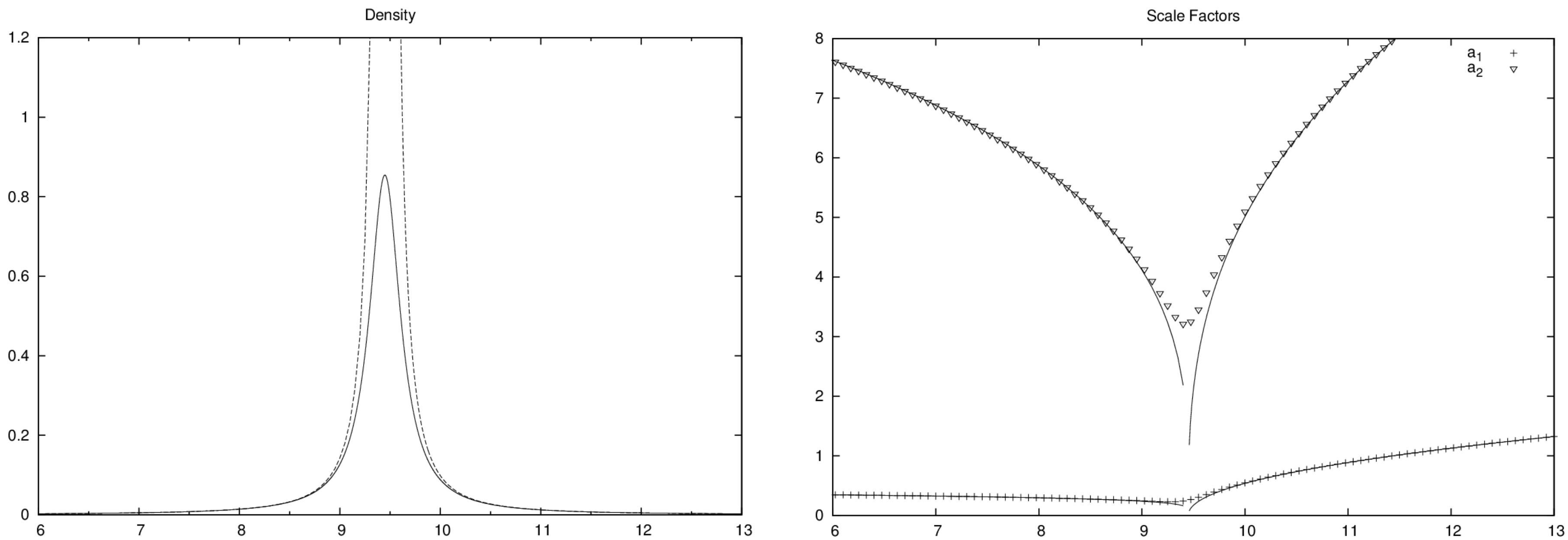

Figure 1: Comparison between classical and effective solutions.

-

-

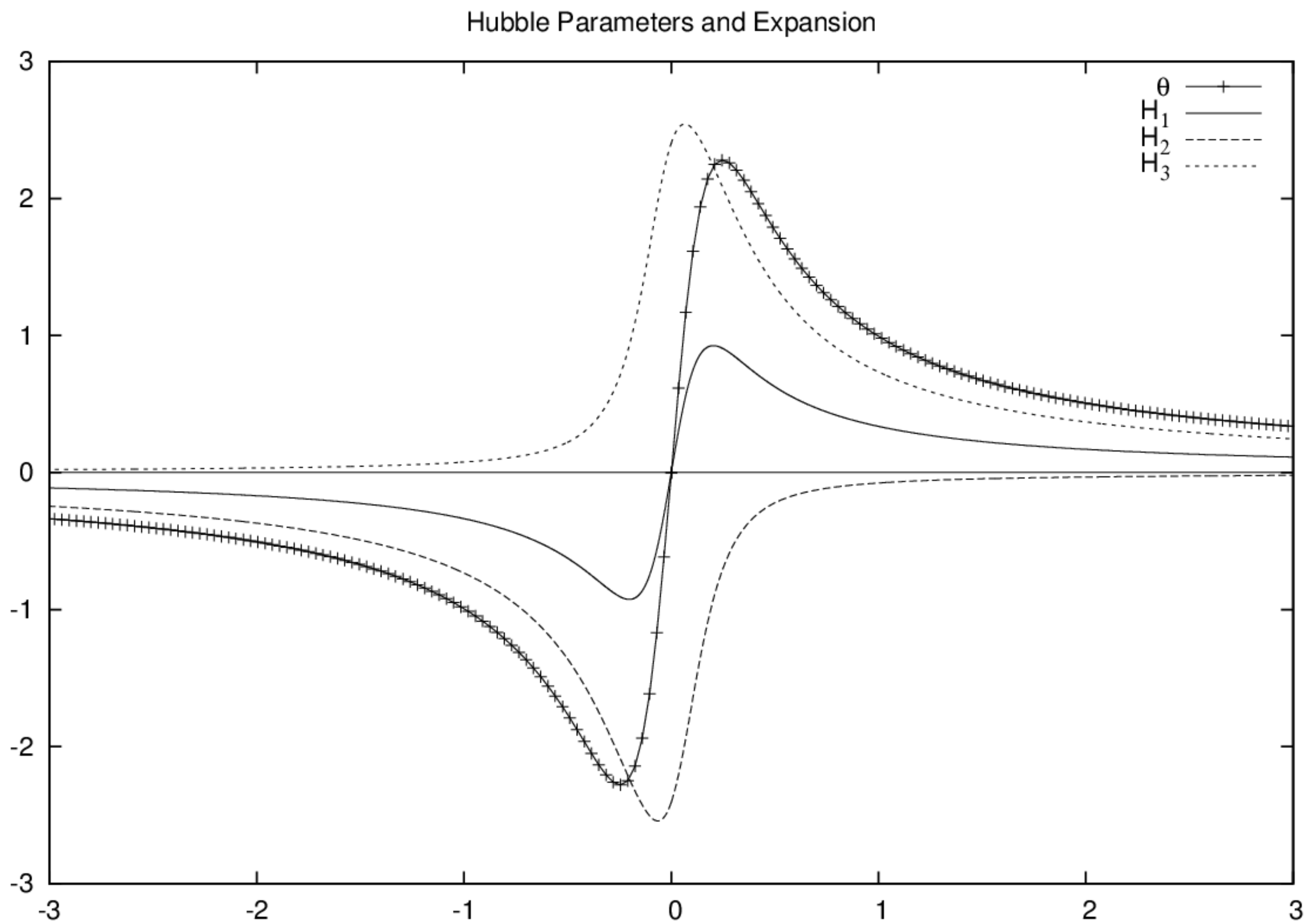

Figure 2: Comparison between classical and effective solutions.

Effective Equations for Bianchi II. General statements.

1. There are solutions where the matter density is larger than the critical density, with point-like and cigar-like singularities.
2. Density can achieve the maximal value ($\rho \approx 0.54\rho_{\rm pl}$) as a consequence of the shear being zero at the bounce and curvature different from zero.
3. Solutions with point like singularity do not saturate the density.
4. Some important solutions are LRS, such as the one with maximal density and the vacuum limit (where all the dynamical contribution comes from the anisotropies).
5. Far away from the bounce, the classical and effective solutions agree.

6. In all the solutions p_ϕ and $\alpha_{32} = c_3 p_3 - c_2 p_2$ are conserved.
7. Bianchi I and therefore the isotropic case are limiting cases of Bianchi II, but they are not contained in Bianchi II.
8. Singularities are resolved: the geodesics are inextendible, i.e., scale factor non zero (or infinite) in a finite time.
9. When $\alpha = 0$ or $x = \sqrt{p_2 p_3 / p_1^3} \to 0$, Bianchi II reduces to Bianchi I.
10. There is one global bounce $\theta = 0$.
11. The Kasner exponents are good quantities to determine which kind of solution are obtained.
12. The shear can be zero at the bounce and non zero in evolution.
13. If p_i bounces a_i not necessarily bounces, it is necessary that $H_i = 0$ for the bounce in direction a_i.

Bianchi IX?

Much more involved. Chaotic in classical theory, with Bianchi I stages with Bianchi II transitions as one approaches the singularity. What will happen when loop quantum cosmology corrections are considered?

Several Quantum Theories (due to ambiguity in quantization). What is their behavior?

Preliminary explorations: Bounce seems to be generic, and the chaotic phase is not seen.

For details, see poster!!

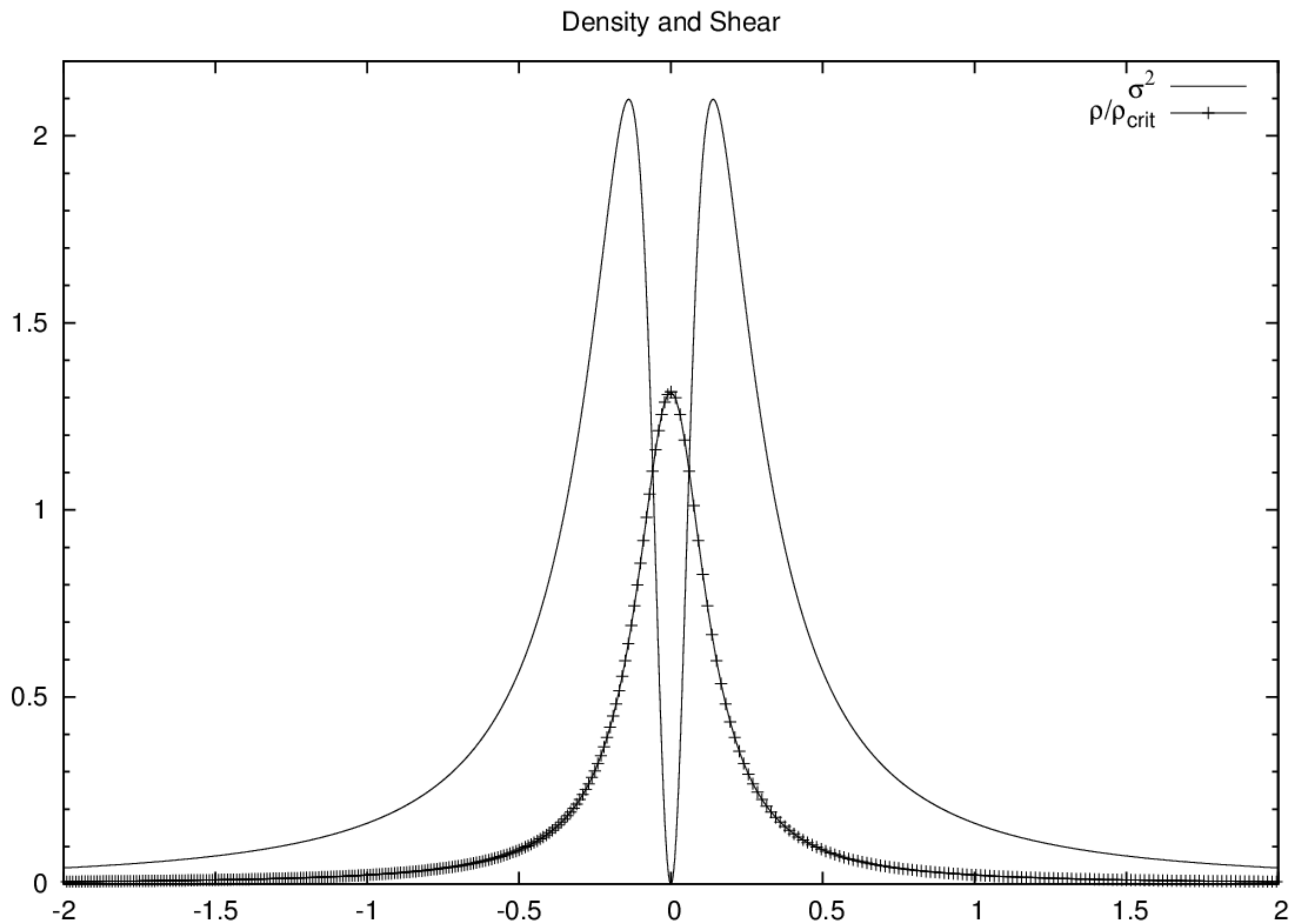

Figure 3: LRS solutions.

Conclusions.

- Loop quantum cosmology employs a completely inequivalent representation than the Schroedinger one (polymeric), motivated by LQG.

- In $k = 0$ LQC, its natural quantum representation seems to be fundamentally different. LQC is intrinsically inequivalent to WDW.

- Full control on semiclassical sector. GR *is* recovered in the appropriate limit.

- For isotropic models, inclusion of spatial curvature $k = \pm 1$, cosmological constant Λ and potential gives similar qualitative picture. (Ashtekar *et al*

- Detailed analysis of Effective description of Bianchi I, Bianchi II and Bianchi IX, provides generic paradigm of singularity resolution in anisotropic models.
- A step toward generic quantum singularity resolution?