Geometric inequalities for black holes

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Geometric inequalities

Geometric inequalities have an ancient history in Mathematics. A classical example is the **isoperimetric inequality** for closed plane curves given by

$$L^2 \geq 4\pi A,$$

where A is the area enclosed by a curve C of length L, and where equality holds if and only if C is a circle.

Geometrical inequalities in General Relativity

- General Relativity is a **geometric theory**, hence it is not surprising that geometric inequalities appear naturally in it. Many of these inequalities are similar in spirit as the isoperimetric inequality.
- However, General Relativity as a **physical theory** provides an important extra ingredient. It is often the case that the quantities involved have a clear physical interpretation and the expected behavior of the gravitational and matter fields often suggest geometric inequalities which can be highly non-trivial from the mathematical point of view.
- The interplay between geometry and physics gives to geometric inequalities in General Relativity their distinguished character.

Well known examples

- **Positive mass theorem:**

$$0 \leq m,$$

with equality if and only if the spacetime is flat.

- **Penrose inequality:**

$$\sqrt{\frac{A}{16\pi}} \leq m,$$

where A is the area of the black hole horizon and the equality holds only for the Schwarzschild black hole.

Angular momentum: the role of axial symmetry

Axial symmetry allows to include angular momentum in the geometrical inequalities for two main reasons:

- For global inequalities is the conservation of angular momentum implied by axial symmetry which is relevant.
- For quasi-local inequalities is the very definition of quasi-local angular momentum (only possible in axial symmetry) which is important.
- These two properties are closed related for vacuum spacetimes, since the Komar integral provides both the conservation law and the definition of quasi-local angular momentum.

Global inequality: mass, angular momentum and charge

Dain 06 08, Chrusciel-Li-Weinstein 08, Chrusciel-Lopes Costa 09, Lopes Costa 10, Schoen-Zhou 11 (in progress).
As a sample of the most general result available we present the following theorem proved by Lopes Costa 10:

Theorem

Consider an axially symmetric, electro-vacuum, asymptotically flat and maximal initial data set with two asymptotics ends. Let m, J and q denote the total mass, angular momentum and charge respectively at one of the ends. Then, the following inequality holds

$$m^2 \geq \frac{q^2 + \sqrt{q^4 + 4J^2}}{2}.$$

Quasi-local inequality: area, angular momentum and charge

Dain 10, Aceña-Dain-Gabach 11, Dain-Reiris 11, Jaramillo-Reiris-Dain 11, Gabach-Jaramillo 11, Gabach-Jaramillo-Reiris 12 (in progress).

Theorem

Given an axisymmetric closed marginally trapped surface $\mathcal{S}$ satisfying the (axisymmetric-compatible) spacetime stably outermost condition, in a spacetime with non-negative cosmological constant and fulfilling the dominant energy condition, it holds the inequality

$$A \geq 4\pi\sqrt{q^4 + 4J^2},$$

where A, J and q are the area, angular momentum and charge of $\mathcal{S}$. If equality holds, then $\mathcal{S}$ is a section of a non-expanding horizon with the geometry of extreme Kerr-Newman throat sphere.

This theorem does not assume vacuum and **does not assume the existence of a maximal slice**.

The size of things: a conjecture

Consider an asymptotically flat, axially symmetric, initial data which satisfy the energy conditions and have $\mathbb{R}^3$ topology. Let Ω be an arbitrary region on the data. Then the following inequality holds:

$$A(\partial\Omega) \geq \frac{8\pi G}{c^3}|J(\Omega)|,$$

where A is the area of the boundary $\partial\Omega$ and J is the angular momentum of the region Ω.

That is: **this inequality is universal for all objects.**

Evidences in favor:

- Neugebauer-Meinel stationary disk exact solution.
- Numerical evidences on initial data (in collaboration with Omar Ortiz).
- Heuristic arguments: the inequality is a consequence of the following basic principles
  - (i) The speed of light c is the **maximum speed**.
  - (ii) If the object has a lot of energy concentrated in a small radius then it is a **black hole**.
  - (iii) The inequality holds for axially symmetric dynamical black holes (previous result).

Physical relevance: if it is true, it is a prediction of the theory than can be verified experimentally.

For more details see the review article:

Geometric inequalities for axially symmetric black holes
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