Loop Quantum Gravity in terms of spinors and harmonic oscillators

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June, 25th 2012

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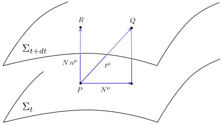
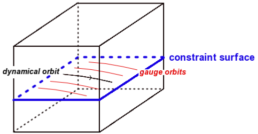
A canonical quantization of General Relativity à la Dirac

- 3+1 space-time splitting
- Dirac's procedure:
  - Representation of the phase space variables as operators in $\mathcal{H}_{\text{kin}}$: $\{.,.\} \rightarrow -i/\hbar\,[.,.]$;
  - Constraints $\mathcal{C} \rightarrow \hat{\mathcal{C}} \in \mathcal{H}_{\text{kin}}$;
  - $|\psi\rangle \,/\, \hat{\mathcal{C}}|\psi\rangle = 0 \rightarrow \mathcal{H}_{\text{phys}}$ + inner product.
- ⚠Constraints in General Relativity: space-time diffeomorphism H^μ
- Ashtekar-Barbero variables: connection A^i_a, triad E^a_i

$$\{A^i_a(x), E^b_j(y)\} = \gamma \delta^i_j \delta^b_a \delta^{(3)}(x, y)$$

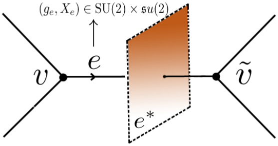




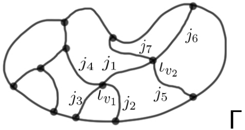
$\rightarrow$ "simple expression" for H^μ + additional $\mathrm{SU}(2)$ Gauss constraint G.

Loop Quantum Gravity kinematical space

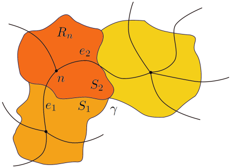
- Holonomy-flux algebra:
  $g_e = \mathcal{P} \exp \int_e A \in \mathrm{SU}(2)$,
  $X_e = \int_{e^*} (gE)^a N_a d^2 S \in \mathfrak{su}(2)$.

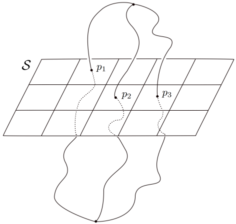


- Basis of $\mathcal{H}_{\mathrm{kin}}$ = spinnetwork states $|\Gamma, j_e, i_v\rangle$

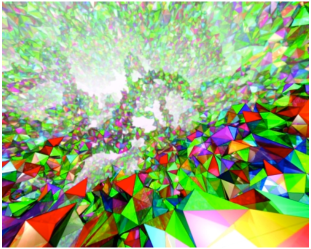


- Diagonalize area/volume operators



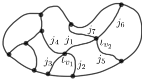


Loop Quantum Gravity: kinematical Hilbert space $\mathcal{H}_{\mathrm{LQG}}$



$\mathcal{H}_{\mathrm{kin}} =$ "collection of certain Hilbert spaces associated to all possible graphs".

Technical reasons $\rightarrow$ truncature of the full continuum theory to a finite-dimensional Hilbert space



$$\mathcal{H}_\Gamma = L^2(\mathrm{SU}(2)^E, d^E g) \subset \mathcal{H}_{\mathrm{kin}}$$

What are the classical degrees of freedom represented by the spinnetwork functions in $\mathcal{H}_\Gamma$?

$\rightarrow$ Spinorial formalism + applications (Spinfoams, Quantum Groups)

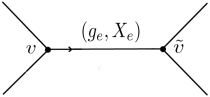
The classical phase space in terms of spinors

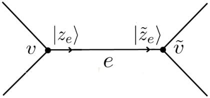
- On a link e

$$\mathrm{SU}(2)\times\mathfrak{su}(2) \simeq T^*\mathrm{SU}(2) \simeq (\mathbb{C}^2\times\mathbb{C}^2)//\mathcal{M}$$

$$(g, X) \qquad |z\rangle = \begin{pmatrix} z^0 \\ z^1 \end{pmatrix} \in \mathbb{C}^2, |\tilde{z}\rangle \in \mathbb{C}^2,$$

$$\mathcal{M} := \langle z|z\rangle - \langle \tilde{z}|\tilde{z}\rangle = 0$$





$$\{X^i, X^j\} = \epsilon^{ijk} X^k \qquad \{z^A, \bar{z}^B\} = -i\delta^{AB}$$

$$\vec{X}(z) =: \frac{1}{2}\langle z|\vec{\sigma}|z\rangle$$

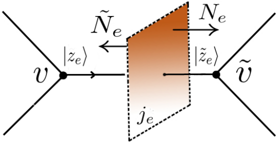
$$g(z,\tilde{z}) = \frac{|z\rangle[\tilde{z}| - |z]\langle\tilde{z}|}{\sqrt{\langle z|z\rangle\langle\tilde{z}|\tilde{z}\rangle}}$$

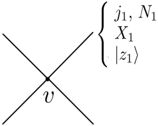
$$\mathcal{M} := \langle z|z\rangle - \langle\tilde{z}|\tilde{z}\rangle = |\vec{X}(z)| - |\vec{X}(\tilde{z})|$$

Geometrical interpretation: twisted geometries

- spinors $|z\rangle$, $|\tilde{z}\rangle$ + matching condition $\mathcal{M}$

$$\rightarrow j, N, \tilde{N}, \xi \text{ s.t. } \begin{cases} j = \frac{\langle z|z\rangle}{2}, \vec{X}(z) = jN, \vec{X}(\tilde{z}) = j\tilde{N}, \\ \xi = -2(\arg z^1 - \arg \tilde{z}^1) \end{cases}$$





- On a given graph Γ: $\times_e T^*\mathrm{SU}(2)//\mathrm{SU}(2)^V \rightarrow$ gauge invariance at each node

$$\mathcal{C} := \sum_{e\in v} X_e = \sum_{e\in v} j_e N_e = 0 \;\rightarrow\; \text{closed twisted geometry}$$

Loop Quantum Gravity/Regge Calculus

Spinor (twistor) space $\mathbb{C}^2 \times \mathbb{C}^2$

$\downarrow$ area matching reduction $\mathcal{M} = 0$

Twisted geometries $(j, N, \tilde{N}, \xi)$

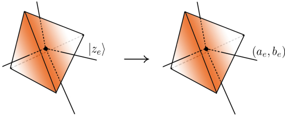
$\downarrow$ closure reduction $\mathcal{C} = 0$

Closed twisted geometries $\rightarrow$ Loop Gravity on a fixed graph

$\downarrow$ shape matching reduction

Regge phase space $\rightarrow$ Regge Calculus (discrete gravity).

Tensor operators and a $\mathrm{U}(N)$ framework



- On a given vertex, quantization scheme

$$z_e^0 \to a_e,\ \bar{z}_e^0 \to a_e^\dagger,$$
$$z_e^1 \to b_e,\ \bar{z}_e^0 \to b_e^\dagger,$$
$$[a_e, a_e^\dagger] = 1 = [b_e, b_e^\dagger],\ [a_e, b_e] = 0.$$

Tensor operators

$$T_e^{1/2} = \begin{pmatrix} a_e^\dagger \\ b_e^\dagger \end{pmatrix},\ \tilde{T}_e^{1/2} = \begin{pmatrix} b_e \\ -a_e \end{pmatrix}$$

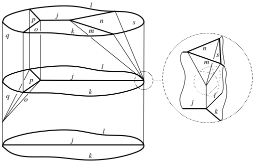
- For each leg e of the vertex, Schwinger Jordan realization of $\mathfrak{su}(2)$: $\hat{X}_e^+ = a_e^\dagger b_e$, $\hat{X}_e^- = a_e b_e^\dagger$, $\hat{X}_e^z = \frac{1}{2}(a_e^\dagger a_e - b_e^\dagger b_e)$
  - Commutation relations : ok
  - $j_e = \frac{1}{2}(n_a + n_b)$, $m_e = \frac{1}{2}(n_a - n_b)$
- $\mathrm{U}(N)$ structure: Operators acting in the intertwiner space (= rank 0 T.O.)

$$E_{ef} = a_e^\dagger a_e + b_e^\dagger b_e,\ [E_{ef}, E_{kl}] = \delta_{fk} E_{el} - \delta_{el} E_{kf},$$
$$F_{ef} = a_e b_f - a_f b_e,\ F_{ef}^\dagger = a_e^\dagger b_f^\dagger - a_f^\dagger b_e^\dagger$$

The space of (N-valent) intertwiners carries an irreps of $\mathrm{U}(N)$.

Beyond the $\mathrm{U}(N)$ framework

- Twistors and covariant twisted geometries: extension to the Lorentz group.
- Spinfoam models: powerful tool to treat the simplicity constraints (second class) which turn BF to gravity
  $\longrightarrow$ Holomorphic spinfoam model:
  amplitude = discrete action in terms of spinors and holonomies.
  Correlation functions = Gaussian integrals



- Quantum Groups: $U_q(\mathfrak{su}(2)) \longrightarrow$ Cosmological constant in LQG? (see talk F. Girelli on Wednesday (16.45))

Conclusions

- Description of the classical phase space of Loop Quantum Gravity in terms of spinors:
  → the standard holonomy-flux structure derived from a simple collection of spinors on a graph (= spinor network)
- Linked to a discrete geometric picture: Twisted Geometries, and to a $\mathrm{U}(N)$ symmetry.
- Applications: news calculation tools,
  e.g. Correlation function using the holomorphic spinfoam amplitude = Gaussian integral.
- Generalization to the gauge group $\mathrm{U}_q(\mathfrak{su}(2))$ thanks to tensor operator techniques.

References

- Twisted geometries: L. Freidel, S. Speziale arXiv:1001.2748, arXiv:1006.0199
- $\mathrm{U}(N)$ framework: F. Girelli, E. Livine arXiv: 0501075; L. Freidel, E. Livine arXiv:0911.3553, arXiv:1005.2090
- Holomorphic Spinfoam models: M.D., E. Livine arXiv: 1006.5666, arXiv: 1104.3683; M.D, L. Freidel, E. Livine, S. Speziale arXiv: 1107.5274
- Spinor representation: E. Livine, J. Tambornino arXiv: 1105.3385; Twistors E. Livine, S. Speziale, J. Tambornino arXiv: 1108.0369
- Review: M.D., S. Speziale, J. Tambornino arXiv:1201.2120
- Quantum Group: M.D., F. Girelli, work in progress (see talk F. Girelli on Wednesday (16.45)).