PROBING THE SPACETIME STRUCTURE THROUGH DYNAMICS

Felipe Tovar Falciano

CBPF - Brazilian Center for Physics Research- Brazil

E-mail: ftovar@cbpf.br

collaborators: E. Goulart, M. Novello and J. D. Toniato

Abstract

We propose to review the meaning of geometrization in view of the analogue program where the emergence of a metric appears as a consequence of linear perturbations. We can show that the self-interacting of a field can be geometrized together with its perturbations in the sense that both dynamics are controlled by the same metric. In the attempt to disentangle the dynamics from the spacetime structure, we have run into a new symmetry of the Klein-Gordon equation that is related to redefinitions of the metric tensor which implement a map between non-equivalent manifolds.

Brief review - effective metrics

Non-linear Scalar Field

- relativistic real scalar field in Minkowski spacetime $\gamma_{\mu\nu}$

$$S=\int L(\varphi,w)\sqrt{-\gamma}\,d^4x$$

$w\equiv\gamma^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi$ is the kinetic term.
$\gamma=det(\gamma_{\mu\nu})$ is the determinant of the metric.

- dynamical equation

$$\left(L_w\partial_\nu\varphi\gamma^{\mu\nu}\right)_{;\,\mu}=\frac{1}{2}L_\varphi \qquad L_w\equiv\frac{dL}{dw}\,,\quad L_\varphi\equiv\frac{dL}{d\varphi}$$

- a quasi-linear second order partial differential equation for φ

$$\hat{g}^{\alpha\beta}(x,\varphi,\partial\varphi)\,\partial_\alpha\partial_\beta\varphi+F(x,\varphi,\partial\varphi)=0,$$

direct calculation

$$\hat{g}^{\mu\nu}\equiv L_w\gamma^{\mu\nu}+2L_{ww}\partial^\mu\varphi\partial^\nu\varphi\quad, \tag{1}$$

determines the principal part (higher order derivatives), $\hat{g}^{\alpha\beta}$ shall be associated with the effective metric.

Geometrical Optics

Eikonal Approximation

- well defined trajectories
- describes the evolution of the characteristic surfaces
- Given the solution φ_0, we construct a family of approximative solutions

$$\varphi(x)=\varphi_0(x)+\alpha\,f(x)e^{iS(x)/\alpha}\qquad,\quad\alpha\in\Re$$

- disperse relation in the limit of fast oscillation ($\alpha\to0$)

$$\hat{g}^{\alpha\beta}\Big|_{\varphi_0}k_\alpha k_\beta=0\quad,\qquad k_\mu\equiv S_{,\,\mu} \tag{2}$$

Eikonal equation determines the causal structure of the theory.

- if $\hat{g}^{\alpha\beta}$ is invertible $\exists\quad\hat{g}_{\mu\nu}$
- defining the affine structure such that $\hat{g}_{\alpha\beta||\nu}=0$

$$\hat{g}^{\mu\nu}k_{\alpha||\mu}k_\nu=0\quad,\qquad\text{geodesic} \tag{3}$$

obs: equations (2) and (3) are conformally invariant

Wave propagation

- describes the small perturbations of a given solution

$$\varphi=\varphi_0+\delta\varphi\qquad\text{with}\quad\delta\varphi^2\ll\delta\varphi$$

- if φ e φ_0 are both solution $\Rightarrow$

$$\Box_{\hat{f}}\,\delta\varphi+m_{eff}^2\delta\varphi=0$$

$$\hat{f}^{\mu\nu}\Big|_{\varphi_0}=L_w^{-2}(1+\beta w)^{-1/2}\hat{g}^{\mu\nu}\quad,\qquad\beta\equiv2L_{ww}/L_w$$

$$m_{eff}^2\Big|_{\varphi_0}\equiv L_w^{-2}(1+\beta w)^{-1/2}\left[L_{\varphi\varphi w}w-\frac{1}{2}L_{\varphi\varphi}+\frac{\partial\hat{g}^{\alpha\beta}}{\partial\varphi}\varphi_{,\alpha;\beta}\right]$$

obs: notation

$$\Box_{\hat{f}}\Psi(x)\equiv\frac{1}{\sqrt{-\hat{f}}}\partial_\nu\left(\sqrt{-\hat{f}}\,\hat{f}^{\mu\nu}\partial_\mu\Psi(x)\right)$$

Effective metrics

- are purely kinematics
- only for the perturbations (waves and rays)
- background behaves as a medium that defines $\hat{g}^{\mu\nu}$
- background dynamics is in $\gamma_{\mu\nu}$

- *Is it possible to geometrize the dynamics of the background scalar field?*
- *Do exist a unique emergent metric for both background $\oplus$ perturbation?*

example: *linear theories in Minkowski*

Emergente Metrics

Classical Quantum Gravity **28**, 245008 (2011)

Theorem 1. *Any scalar non-linear theory described by the lagrangian $L(w,\varphi)$ is equivalent to the field φ propagating in an emergent spacetime with metric $\hat{h}_{\mu\nu}(\varphi,\partial\varphi)$ and a suitable source $j(\varphi,\partial\varphi)$, both constructed explicitly in terms of the field and its derivatives. Furthermore, in the optical limit, the wave vectors associated with its perturbations follow null geodesics in the same $\hat{h}_{\mu\nu}(\varphi,\partial\varphi)$ metric.*

No explicit dependence in φ, i. e. $L(w)$

$$\frac{1}{\sqrt{-\gamma}}\partial_\mu\left(\sqrt{-\gamma}\,L_w\partial_\nu\varphi\gamma^{\mu\nu}\right)=0$$

Let us consider $\hat{h}^{\mu\nu}(\varphi,w,\gamma)=A\left(\gamma^{\mu\nu}+B\varphi^{,\mu}\varphi^{,\nu}\right)$

As a direct consequence

$$\hat{h}^{\mu\nu}\partial_\nu\varphi=A(1+Bw)\gamma^{\mu\nu}\partial_\nu\varphi\quad,\quad\sqrt{-\hat{h}}=A^{-2}(1+Bw)^{-1/2}\sqrt{-\gamma}$$

Requiring that $\sqrt{-\hat{h}}\,\hat{h}^{\mu\nu}\partial_\nu\varphi=\sqrt{-\gamma}\,L_w\partial_\nu\varphi\gamma^{\mu\nu}\quad\Rightarrow\quad A=\frac{\sqrt{1+Bw}}{L_w}$

In other to fix B $\quad\hat{h}^{\mu\nu}\propto\hat{g}^{\mu\nu}\quad\Rightarrow\quad B=\beta=2\frac{L_{ww}}{L_w}$

$$\hat{h}^{\mu\nu}\equiv\frac{\sqrt{1+\beta w}}{L_w}(\gamma^{\mu\nu}+\beta\varphi_{,\mu}\varphi_{,\nu})\quad\text{or}\quad\hat{h}_{\mu\nu}\equiv\frac{L_w}{\sqrt{1+\beta w}}\left(\gamma_{\mu\nu}-\frac{\beta}{1+\beta w}\varphi_{,\mu}\varphi_{,\nu}\right)$$

In this manner, **it is equivalent!**

Non-linear in Flat ST
$$\begin{cases}\partial_\mu\left(L_w\partial_\nu\varphi\gamma^{\mu\nu}\right)=0\\ \gamma^{\alpha\beta}k_\alpha k_\beta=0\\ \gamma^{\mu\nu}k_{\alpha;\mu}\,k_\nu=0\end{cases}$$
, Linear in Curved ST
$$\begin{cases}\Box_{\hat{h}}\varphi=0\\ \hat{h}^{\alpha\beta}k_\alpha k_\beta=0\\ \hat{h}^{\mu\nu}k_{\alpha||\mu}k_\nu=0\end{cases}$$

- non-linear theories in Minkowski $\sim$ "free" Klein-Gordon in curved ST
- a single emergent metric defined for background and its perturbations
- **including φ, it appears a source** $j(\varphi,\partial\varphi)\equiv\frac{L_\varphi}{2L_w^2}(1+\beta w)^{3/2}$
- $L(w,\varphi)=w+V(\varphi)$ **trivialize to Minkowski**

Back-reaction

- In straight analogy to GR, a singe solution is a pair $(\varphi_0\,,\hat{h}_0^{\mu\nu})$
  given $\varphi_1=\varphi_0+\delta\varphi\quad\Rightarrow\quad\hat{h}_1^{\mu\nu}=\hat{h}_0^{\mu\nu}+\delta\hat{h}^{\mu\nu}$
  If $\quad\delta\varphi^2<<\delta\varphi\quad\Rightarrow\quad\begin{cases}\Box_{\hat{f}}\delta\varphi+m^2\delta\varphi=0\\ \hat{f}^{\mu\nu}=\left[(1+\beta w)^{-1}\hat{h}^{\mu\nu}\right]_{\varphi_0}\end{cases}$
- agrees with $\hat{f}^{\mu\nu}$analogue models linear approximations

Hydrodynamical flow and Newtonian approximation

- Theory $L(w,\varphi)$ with $w>0\quad\Rightarrow\quad$ barotropic flow without rotations
- potential velocity of a barotropic fluid evolves in the emergent metric

$$\hat{h}^{\mu\nu}=\frac{\mu}{2nc_s}\left[\gamma^{\mu\nu}+(c_s^{-2}-1)v^\mu v^\nu\right]$$

- taking the limits $v^\mu\longrightarrow v^\mu=(1,\vec{v})\quad,\quad|\vec{v}|<<1$

Recover known results of analogue models
fluid's excitations

$$\hat{f}^{\mu\nu}=\frac{1}{2\rho c_s}\begin{pmatrix}1 & \vdots & v^j\\ \cdots & \cdot & \cdots\\ v^i & \vdots & (-c_s^2\delta^{ij}+v^iv^j)\end{pmatrix}$$

- Non-linear theory is mapped into linear theory in curved ST
- What happens with linear theories?...
  Is it possible to map the linear Klein-Gordon Theory in Minkowski ST in something else?

Metric Degeneracy and Algebraic Symmetry

Classical Quantum Gravity **29**, 085011 (2012)

Let a manifold $\mathcal{M}$ with $D=1+d$ with metric $\gamma_{\mu\nu}$

$$\Box\varphi\equiv\frac{1}{\sqrt{-\gamma}}\partial_\mu\left(\sqrt{-\gamma}\gamma^{\mu\nu}\partial_\nu\varphi\right)=0$$

Defining the tensor $q^{\mu\nu}\equiv A\,\gamma^{\mu\nu}+B\,\gamma^{\mu\alpha}\gamma^{\nu\beta}\partial_\alpha\varphi\,\partial_\beta\varphi$

<u>case $w\neq0$</u>

Its inverse is $\quad q_{(A)\mu\nu}=\frac{1}{A}\gamma_{\mu\nu}-\frac{B}{A(A+Bw)}\partial_\mu\varphi\partial_\nu\varphi$

$$\sqrt{-q_{(A)}}=\sqrt{-\gamma}\left[A^{-d/2}(A+Bw)^{-1/2}\right]\quad,\quad q_{(A)}^{\mu\nu}\partial_\nu\varphi=(A+Bw)\gamma^{\mu\nu}\partial_\nu\varphi$$

by choosing... $\sqrt{-q_{(A)}}\,q_{(A)}^{\mu\nu}\partial_\nu\varphi=\sqrt{-\gamma}\,\gamma^{\mu\nu}\partial_\nu\varphi\quad\Rightarrow\quad B=(A^d-A)/w$

$$q_{(A)\mu\nu}=\frac{1}{A(x)}\left(\gamma_{\mu\nu}-\frac{1-A(x)^{1-d}}{w}\partial_\mu\varphi\partial_\nu\varphi\right)$$

<u>case $w=0$</u>

Its inverse is $\quad q_{(A)\mu\nu}=\frac{1}{A}\gamma_{\mu\nu}-\frac{B}{A^2}\partial_\mu\varphi\partial_\nu\varphi$

$$\sqrt{-q_{(A)}}=\sqrt{-\gamma}A^{-(1+d)/2}\quad,\quad q_{(A)}^{\mu\nu}\partial_\nu\varphi=A\gamma^{\mu\nu}\partial_\nu\varphi$$

by choosing... $\sqrt{-q_{(A)}}\,q_{(A)}^{\mu\nu}\partial_\nu\varphi=\sqrt{-\gamma}\,\gamma^{\mu\nu}\partial_\nu\varphi\quad\Rightarrow\quad A=1$

$$q_{(A)\mu\nu}=\gamma_{\mu\nu}-A(x)\,\partial_\mu\varphi\partial_\nu\varphi$$

$$\Box_{q_{(A)}}\varphi\;=\;\Box_\gamma\varphi\;=\;0$$

- $A(x)$ is completely arbitrary non-zero function
- action $S=\int d^4x\sqrt{-\gamma}\,\gamma^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi$ is also invariant
- metricity $\nabla^\mu_{(A)}q^{\alpha\beta}_{(A)}=0\quad\Rightarrow\quad\nabla_{(A)\,\mu}T^{\mu\nu}_{(A)}=0$
- $\exists$ multiples metrics for the same dynamics.

Group structure

- Defining the transformation $\mathcal{T}_a\,[\gamma^{\mu\nu}]\equiv q^{\mu\nu}_{(a)}$

$$\mathcal{T}_b\,[q^{\mu\nu}_{(a)}]=\gamma^{\mu\nu}+(a+b)\gamma^{\mu\alpha}\gamma^{\nu\beta}\partial_\alpha\varphi\partial_\beta\varphi\qquad w=0$$

$$=b.a\left\{\gamma^{\mu\nu}+\frac{(b.a)^{d-1}-1}{w}\gamma^{\mu\alpha}\gamma^{\nu\beta}\partial_\alpha\varphi\partial_\beta\varphi\right\}\qquad w\neq0$$

- Properties

i) Identity $\mathcal{T}_1\circ\mathcal{T}_A=\mathcal{T}_A\circ\mathcal{T}_1=\begin{cases}\mathcal{T}_1=\mathcal{T}_{A^{-1}} & w\neq0\\ \mathcal{T}_1=\mathcal{T}_{A=0} & w=0\end{cases}$

ii) Closure $\quad\mathcal{T}_B\circ\mathcal{T}_A=\begin{cases}\mathcal{T}_{(BA)} & w\neq0\\ \mathcal{T}_{(B+A)} & w=0\end{cases}$

iii) Inverse $\quad\mathcal{T}_A^{-1}\circ\mathcal{T}_A=\mathcal{T}_A\circ\mathcal{T}_A^{-1}=\mathcal{T}_1\quad\begin{cases}\mathcal{T}_A^{-1}=\mathcal{T}_{(A^{-1})} & w\neq0\\ \mathcal{T}_A^{-1}=\mathcal{T}_{(-A)} & w=0\end{cases}$

iv) Associativity $\quad\mathcal{T}_C\circ\left(\mathcal{T}_B\circ\mathcal{T}_A\right)=\mathcal{T}_C\circ\mathcal{T}_{(BA)}=\mathcal{T}_{(CBA)}=\left(\mathcal{T}_C\circ\mathcal{T}_B\right)\circ\mathcal{T}_A$

- an abelian infinite parameter group

General Remarks

- Generalizations
  1. complex scalar field $\sim$ two non-coupled real scalar fields
  2. electromagnetic fields
- Issues
  1. What happens if we consider gravitational fields?
  2. many metrical structures (infinite!) associated with the same dynamics
  3. Are these metrics solutions to Einstein's fileds equations?

References

[1] E Goulart et al., *Class. Quant.Grav.* **28**, 245008 (2011).

[2] F T Falciano and E Goulart, *Class. Quant.Grav.* **29**, 085011 (2012).