Geometric operators in loop quantum gravity with a cosmological constant

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Work in progress, in collaboration with Maite Dupuis

Quantum gravity and cosmological constant

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  - $\Lambda = 0$: Ponzano Regge model (divergent amplitude) based irreps of *su(2)* or *so(2,1)*.
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* **Quantum gravity in 4d:**

  - $\Lambda = 0$: EPRL model (divergent amplitudes) based on *so(4)* or *so(3,1)*.
  - $\Lambda \neq 0$: EPRL model (finite amplitude) constructed from $\mathcal{U}_q(so(3,1))$ $\quad q = e^{-\ell_p^2/\ell_c^2}$

Quantum gravity and cosmological constant

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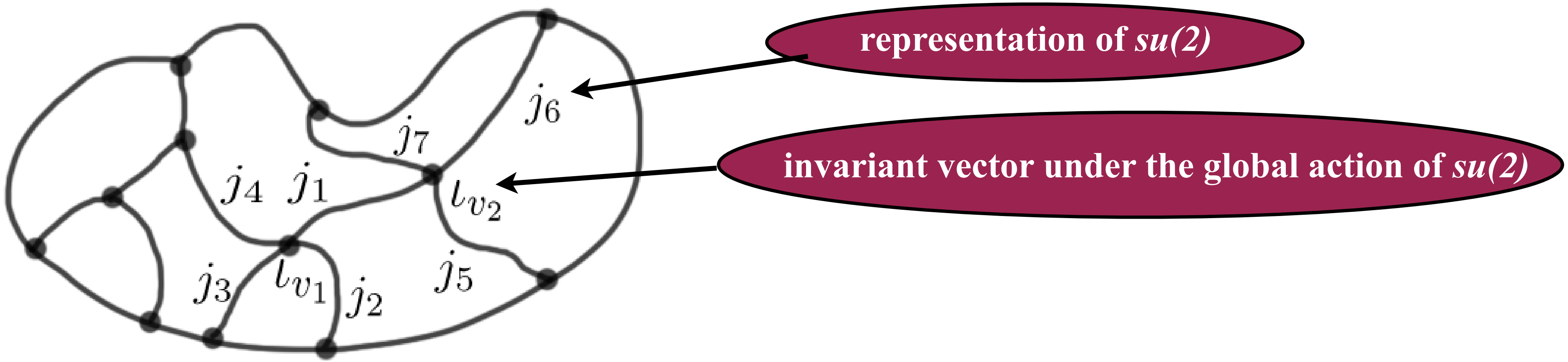
Can we try to better understand the notion of quantum geometry (in the LQG context) built from a quantum group?

- Strangely (*or not*), not much work done in this direction: only papers by **S. Major/L. Smolin in 95**.
- Hopefully we can understand better why a quantum group encodes the notion of cosmological constant.

Quantum geometry in LQG with no cosmological constant

* Fundamental piece of quantum space is given by **intertwinner**.



* **Geometric (kinematic) observables: space geometry is quantized**
- ☆ *Length* (in 2d space) or *Area* (in 3d space) has discrete spectrum.
- ☆ *Cosine of angle* has discrete spectrum. cf talk by S. Major
- ☆ *Volume* has discrete spectrum.
- ☆ *Algebra of kinematical observables* generated by a *U(N)* algebra built from Schwinger-Jordan trick.

Dupuis, Freidel, Girelli, Livine,...

These operators are invariant under the adjoint action of su(2)

Al Kashi's rule

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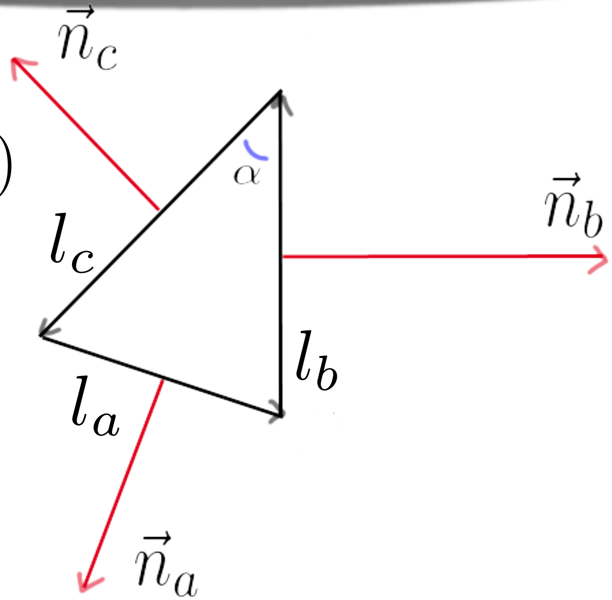
$$\vec{n}_b \cdot \vec{n}_c = l_b l_c \cos\alpha = \frac{1}{2}\left(l_b^2 + l_c^2 - l_a^2\right)$$



$$\vec{n}_a + \vec{n}_b + \vec{n}_c = \vec{0}$$

$$|\vec{n}_i| = l_i^2$$

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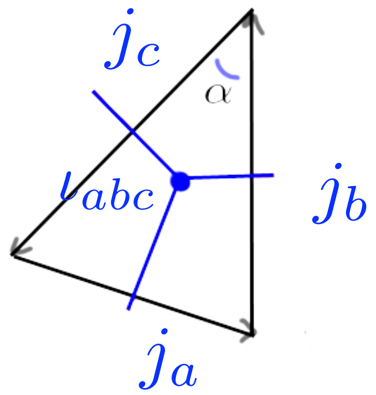

$${}^{(a)}\overrightarrow{J} + {}^{(b)}\overrightarrow{J} + {}^{(c)}\overrightarrow{J} = \vec{0}$$

$${}^{(a)}\vec{J} \cdot {}^{(b)}\vec{J}\,|\iota_{abc}\rangle = ?$$

- In the LQG approach, normals are quantized. One calculates explicitly using the LQG quantization (2+1 space time, ie 2d space):

$$\vec{n}_i \longrightarrow {}^{(i)}\vec{J}, \quad \vec{J} \in su(2)$$

$${}^{(a)}\vec{J} \equiv \vec{J} \otimes 1 \otimes 1, \; {}^{(b)}\vec{J} \equiv 1 \otimes \vec{J} \otimes 1, \ldots$$

$$|\vec{n}|^2 = l^2 \longrightarrow j(j+1)\ell_p$$

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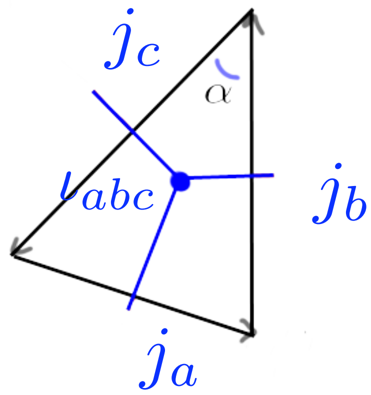

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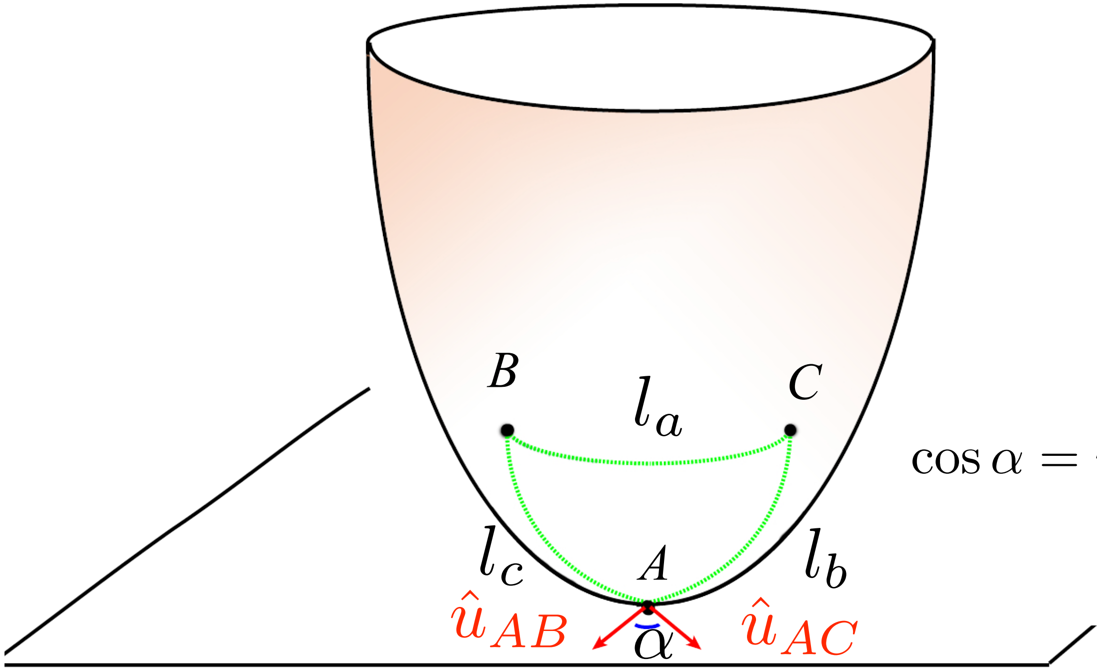
$${}^{(a)}\vec{J} \cdot {}^{(b)}\vec{J}|\iota_{abc}\rangle = \frac{1}{2}\left(j_b(j_b+1) + j_c(j_c+1) - j_a(j_a+1)\right)|\iota_{abc}\rangle$$

* **We have a quantization of Al Kashi's rule!**

Hyperbolic cosine law

* In the case of hyperbolic geometry, Al Kashi's rule is generalized to the hyperbolic cosine law. (Still 2+1 spacetime now with positive Λ)



Radius of hyperboloid is ℓ_c

$$\cos\alpha = \hat{u}_{AB}\cdot\hat{u}_{AC} = \frac{-\cosh\frac{l_a}{\ell_c} + \cosh\frac{l_b}{\ell_c}\cosh\frac{l_c}{\ell_c}}{\sinh\frac{l_b}{\ell_c}\sinh\frac{l_c}{\ell_c}}$$

* **Can we recover some kind of quantization of this law using a quantum group?**

Quantum group

- The main features of quantum group $\mathcal{U}_q(su(2))$ with q real. (*Case with q root of unity is much more involved in terms of representations*.)

- **Algebra** $$[J_z, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = [2J_z]_q, \text{ with } [J_z]_q = \frac{q^{J_z/2} - q^{-J_z/2}}{q^{1/2} - q^{-1/2}}.$$

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☆ **Representations are similar to classical case** $|j, m\rangle$

$$J_\pm |j, m\rangle = \sqrt{[j \mp m]_q [j \pm m + 1]_q}\, |jm \pm 1\rangle$$

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- **Adjoint action on some operator** f

$$J_\pm \triangleright f = J_\pm f q^{-J_z/2} - q^{\pm 1/2} q^{-J_z/2} f J_\pm, \quad J_z \triangleright f := J_z f - f J_z.$$

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- **Consequence, eg** $\quad J_+ \triangleright (J_+) = (q - q^{1/2}) q^{-J_z/2} J_+^2 \neq 0.$

and none of naive operators built from *J are invariant under adjoint action.*

Solution: use tensor operators

q real: ok!
q root of unity: not clear yet

* ***Definition*** (Rittenberg et al) (which works for quasi triangular Hopf algebras)
A tensor operator $\mathbf{t}$ is defined from the intertwinning map

$$\begin{array}{rcl} \mathbf{t}: V & \rightarrow & L(W,W) \\ x & \rightarrow & \mathbf{t}(x) \end{array} \qquad \mathbf{t}(|j,m\rangle) \equiv \mathbf{t}^j_m$$

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✱ ***Realization of some tensor operators***
The Jordan-Schwinger trick provides a realization of spinor operators (*q=1*).

$$T^{1/2} = \begin{pmatrix} a^\dagger \\ b^\dagger \end{pmatrix}, \qquad \tilde{T}^{1/2} = \begin{pmatrix} b \\ -a \end{pmatrix} \qquad J_+ = a^\dagger b \propto \sum_{m_1,m_2} C \begin{matrix} 1/2 & 1/2 & 1 \\ m_1 & m_2 & +1 \end{matrix}\ T^{1/2}_{m_1} \tilde{T}^{1/2}_{m_2} = \mathbf{t}^1_{+1}$$

In the quantum group case, generators J are not vector operators!

Tensor operators and observables

The ***tensor product*** of tensor operators is more complicated to construct in the quantum group case.

* ***Proposition*** (Rittenberg et al)

If $\mathbf{t}$ is a tensor operator then ${}^{(2)}\mathbf{t} \equiv \psi_{\mathcal{R}}(\mathbf{t} \otimes \mathbf{1})\psi_{\mathcal{R}}^{-1}$ is another tensor operator, and $\psi_{\mathcal{R}} = \psi \circ \mathcal{R}$ is the permutation deformed by the $\mathcal{R}$ matrix.

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✱ ***Intertwinner observables from tensor operators***

From spinor operators:

$$E_{ij} \propto \sum_{m_1 m_2} {}_qC \begin{matrix} 1/2 & 1/2 & 0 \\ m_1 & m_2 & 0 \end{matrix} \; {}^{(i)}T^{1/2}_{m_1} \, {}^{(j)}\tilde{T}^{1/2}_{m_2} \rightarrow a_i^\dagger a_j + b_i^\dagger b_j \; for \; q \rightarrow 1$$

Recover the U(n) formalism!

From vector operators:

"Scalar product"

$${}^{(i)}\mathbf{t}^1 \cdot {}^{(j)}\mathbf{t}^1 \equiv \sum_{m_1 m_2} {}_qC \begin{matrix} 1 & 1 & 0 \\ m_1 & m_2 & 0 \end{matrix} \; {}^{(i)}\mathbf{t}^1_{m_1} \, {}^{(j)}\mathbf{t}^1_{m_2}$$

Quantization of cosine and area/length

"Vector product"

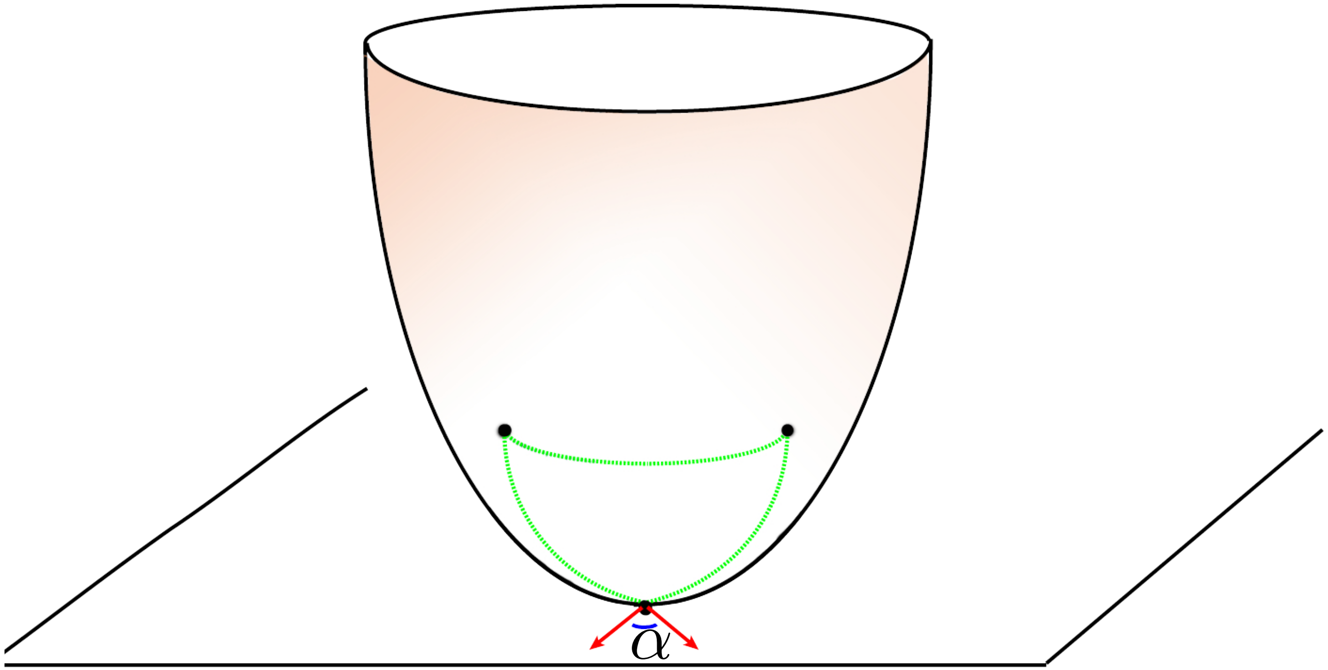
$$\left({}^{(i)}\mathbf{t}^1 \wedge {}^{(j)}\mathbf{t}^1 \right)_M \equiv \sum_{m_1 m_2} {}_qC \begin{matrix} 1 & 1 & 1 \\ m_1 & m_2 & M \end{matrix} \; {}^{(i)}\mathbf{t}^1_{m_1} \, {}^{(j)}\mathbf{t}^1_{m_2}$$

$${}^{(k)}\mathbf{t}^1 \cdot \left({}^{(i)}\mathbf{t}^1 \wedge {}^{(j)}\mathbf{t}^1 \right)$$

Building block for volume operator

Hyperbolic cosine law revisited

* We can calculate explicitly the action of the scalar product on intertwinner.



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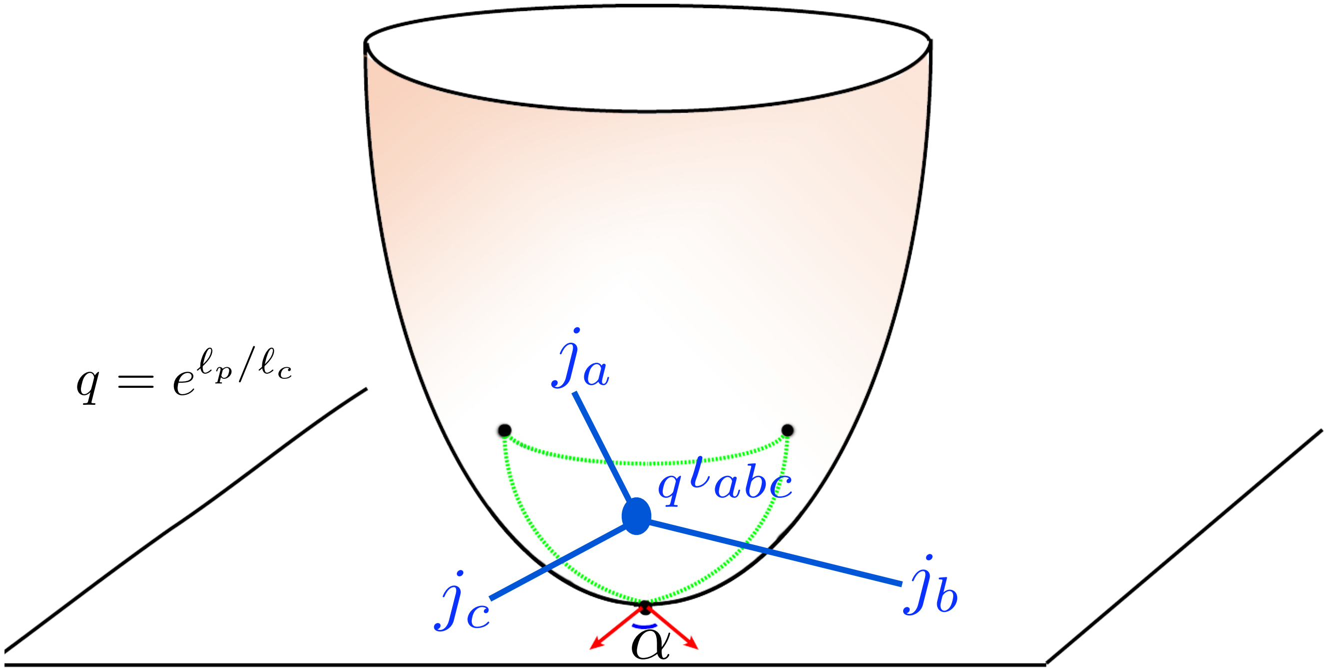

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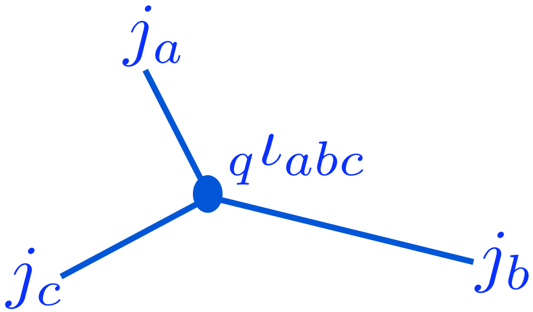

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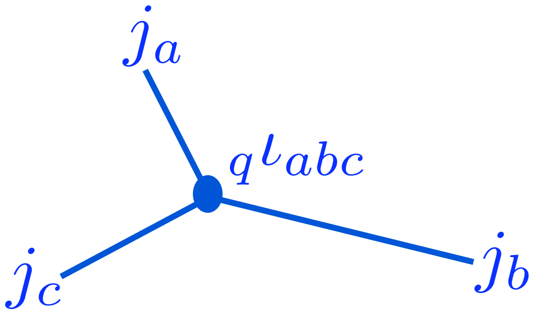

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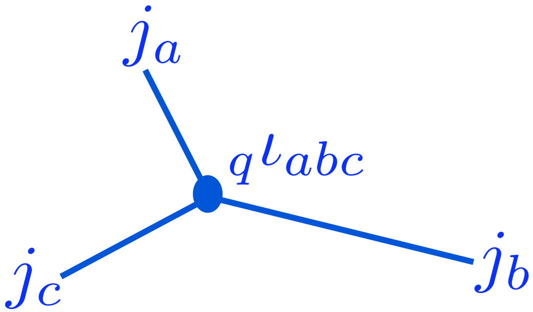

* **We have recovered a quantization of the hyperbolic cosine law!**

$$ \cos\alpha = \frac{-\cosh\frac{l_a}{\ell_c} + \cosh\frac{l_b}{\ell_c}\cosh\frac{l_c}{\ell_c}}{\sinh\frac{l_b}{\ell_c}\sinh\frac{l_c}{\ell_c}} $$

Outlook

* **Main results**:

- Tensor operators are a key-tool to construct (kinematic) observables in loop quantum gravity.

- We are able to construct kinematical observables in the quantum group case.

* **To explore further:**

- The geometric observables built from quantum group have to be studied further (eg the volume operator).

- We have a generalization of the *U(n)* formalism to the quantum group case, however it is not clear yet if this is $\mathcal{U}_q(u(n))$(ie a deformation of the *U(n)* formalism).

- The hamiltonian constraint in 3d can be constructed using the *U(n)* formalism (**Bonzom-Livine**). Hopefully, using the quantum group generalization will provide a better understanding of why a quantum group structure appears due to the presence of the cosmological constant.

- We probably have the right framework to study twisted geometries in the presence of a cosmological constant.