Another perspective on Einstein's "Prague" field-equation of 1912

Domenico Giulini

ZARM Bremen and University of Hannover

Relativity and Gravitation
100 Years after Einstein in Prague
Prague, June 25. 2012

3. *Lichtgeschwindigkeit und Statik des Gravitationsfeldes;*

von A. Einstein.

$$\Delta c = k c \varrho ,$$

1) In einer in kurzem nachfolgender Arbeit wird gezeigt werden, daß die Gleichung (5a) und (5b) noch nicht exakt richtig sein können. In dieser Arbeit sollen sie vorläufig benutzt werden.

Prag, Februar 1912.

(Eingegangen 26. Februar 1912.)

Zur Theorie des statischen Gravitationsfeldes; von A. Einstein.

$$\Delta c = k\left\{c\sigma + \frac{1}{2k}\frac{\mathrm{grad}^2 c}{c}\right\}$$

- In this talk I wish to show how to arrive at this equation on an somewhat naïve but pedagogically appealing route and discuss simple features of (spherically symmetric) solutions that remind us on GR. (Based on D.G., PLA 232 165-170 (1996))

Zur Theorie des statischen Gravitationsfeldes; von A. Einstein.

$$\Delta c = k\left\{c\,\sigma + \frac{1}{2k}\frac{\mathrm{grad}^2 c}{c}\right\}$$

- In this talk I wish to show how to arrive at this equation on an somewhat naïve but pedagogically appealing route and discuss simple features of (spherically symmetric) solutions that remind us on GR.
  (Based on D.G., PLA 232 165-170 (1996))

Newtonian Gravity I

- Field equation

$$\Delta\varphi = 4\pi G\rho \tag{1}$$

- Force per unit volume

$$\vec{f} = -\rho\vec{\nabla}\varphi \tag{2}$$

- Real-time redistribution of mass along flow lines of velocity field $\vec{\xi}$ changes ρ by ($\delta\vec{\xi} := \vec{\xi}\delta t$)

$$\delta\rho = \frac{-L_{\delta\vec{\xi}}(\rho\, d^3x)}{d^3x} = -\vec{\nabla}(\delta\vec{\xi}\rho) \tag{3}$$

and requires work ($B :=$supp(ρ)):

$$\delta A = -\int_{\mathbb{R}^3} \delta\vec{\xi}\cdot\vec{f} = -\int_B \varphi\,\vec{\nabla}(\delta\vec{\xi}\rho) = \int_B \varphi\,\delta\rho \tag{4}$$

Newtonian Gravity II

- Throughout the redistribution process we assume field equations to apply:

$$\Delta\delta\varphi = 4\pi G\,\delta\rho \tag{5}$$

- This leads to

$$\delta A = \int_B \varphi\,\delta\rho = \delta\left\{-\frac{1}{8\pi G}\int_{\mathbb{R}^3}(\vec{\nabla}\varphi)^2\right\} \tag{6}$$

- We identify the energy density ε of the static gravitational field as

$$\varepsilon = -\frac{1}{8\pi G}\,(\vec{\nabla}\varphi)^2 \tag{7}$$

The Principle

- Change the field equation in a way compatible with the principle, that all energies source the gravitational field according to $E = mc^2$.
- The active gravitational mass is defined via its flux at spatial infinity:

$$M_g = \frac{1}{4\pi G} \int_{S^2_\infty} \vec{n} \cdot \vec{\nabla}\varphi = \frac{1}{4\pi G} \int_{\mathbb{R}^3} \Delta\varphi \tag{8}$$

- The Principle takes the form

$$\delta A = \int_B \varphi \, \delta\rho \overset{!}{=} \delta M_g = \frac{1}{4\pi G} \int_{\mathbb{R}^3} \Delta\delta\varphi \tag{9}$$

- The right-hand side will depend on field equation. Newtonian gravity fails the principle since $\delta M_g = 0$ always.

Applying The Principle

- Include Newtonian field energy as source

$$\Delta\varphi = 4\pi G \left(\rho - \frac{1}{8\pi G c^2}\left(\nabla\varphi\right)^2\right) \tag{10}$$

- Use this equation to calculate δM_g:

$$\delta M_g = \int_B \sum_{n=0}^{N-1} \frac{1}{n!}\left(\frac{\varphi}{c^2}\right)^n \delta\rho + \underbrace{\frac{1}{N!c^{2N}}\frac{1}{4\pi G}\int_{\mathbb{R}^3} \varphi^N \delta(\Delta\varphi)}_{\to 0 \text{ for } N\to\infty} \tag{11}$$
$$\to \int_B \delta\rho \, \exp\left(\varphi/c^2\right)$$

- Infer self-consistency if

$$\Phi := c^2 \exp\left(\varphi/c^2\right) \tag{12}$$

rather than φ were potential, so that $\vec{f} = -\rho\vec{\nabla}\Phi$, and (10) is maintained.

Improved Newtonian Theory

- The new field equation uniquely arrived at is

$$\Delta\Phi = \frac{4\pi G}{c^2}\left\{\Phi\rho + \frac{c^2}{8\pi G}\frac{(\vec{\nabla}\Phi)^2}{\phi}\right\} \tag{13}$$

$$\Delta c = k\left\{c\,\sigma + \frac{1}{2k}\,\frac{\mathrm{grad}^2 c}{c}\right\}$$

- It becomes linear if written in terms of $\Psi = c^2\sqrt{\Phi/c^2}$:

$$\Delta\Psi = \frac{2\pi G}{c^2}\rho\Psi \tag{14}$$

A Theorem

For any spherically symmetric solution with $supp(\rho) \subset B_R(0)$ we have

$$\frac{GM_g}{2c^2} < R \tag{15}$$

Mass radii (homogeneous star)

- For a homogeneous star of radius R and mass-density ρ have "baryonic" mass

$$M_b := \frac{4\pi}{3} R^3 \rho \tag{16}$$

- Introduce baryonic and gravitational mass-radii

$$R_g := \frac{GM_g}{2c^2}, \qquad R_b := \frac{GM_b}{2c^2} \tag{17}$$

- Relation between $y := R_g/R$ and $x := R_b/R$ is:

$$y = 1 - \frac{\tanh\sqrt{3x}}{\sqrt{3x}}, \qquad [0, \infty) \to [0, 1) \tag{18}$$

How energies gravitate (homogeneous star)

- Total, matter (baryonic), and field energies gravitate as follows:

$$M_g c^2 = E_{\text{total}} = M_b c^2 \left(1 - \frac{3}{5}\frac{R_b}{R} + \mathcal{O}(x^2)\right) \quad (19)$$

$$E_{\text{field}} = M_b c^2 \left(0 + \frac{3}{5}\frac{R_b}{R} + \mathcal{O}(x^2)\right) \quad (20)$$

$$E_{\text{matter}} = M_b c^2 \left(1 - \frac{6}{5}\frac{R_b}{R} + \mathcal{O}(x^2)\right) \quad (21)$$

- Binding energy as in Newtonian theory.
- Field energy equal in modulus but opposite in sign (now positive!).
- Matter energy is diminished ("redshifted") twice as fast due to appearance of $\rho\Phi$ – rather than just ρ – on r.h.s. of (13).

$\Rightarrow$ Balance OK

THANK YOU!

How energies gravitate (homogeneous star)

- Total, matter (baryonic), and field energies gravitate as follows:

$$M_g c^2 = E_{\text{total}} = M_b c^2 \left(1 - \frac{3}{5}\frac{R_b}{R} + \mathcal{O}(x^2)\right) \tag{19}$$

$$E_{\text{field}} = M_b c^2 \left(0 + \frac{3}{5}\frac{R_b}{R} + \mathcal{O}(x^2)\right) \tag{20}$$

$$E_{\text{matter}} = M_b c^2 \left(1 - \frac{6}{5}\frac{R_b}{R} + \mathcal{O}(x^2)\right) \tag{21}$$

- Binding energy as in Newtonian theory.
- Field energy equal in modulus but opposite in sign (now positive!).
- Matter energy is diminished ("redshifted") twice as fast due to appearance of $\rho\Phi$ – rather than just ρ – on r.h.s. of (13).

$\Rightarrow$ Balance OK

THANK YOU!