Geodesic equations and algebro-geometric methods

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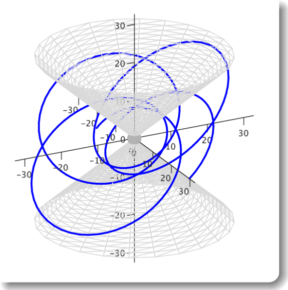

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Space-times and geodesics

- The observation of particles and light allows to conclude on physical properties of the space-time.
- A space-time may be characterized by its complete set of geodesics.

Geodesics and analytical solutions

- The complete set of geodesics can best be explored by using analytical methods
- Analytical solutions for geodesics in a wide range of space-times can be found by algebro-geometric methods

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Geodesic equation

- describes the motion of massive test particles and light

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{ds}\frac{dx^\sigma}{ds} = 0$$

where $\Gamma^\mu_{\rho\sigma} = \frac{1}{2}g^{\mu\alpha}(\partial_\rho g_{\sigma\alpha} + \partial_\sigma g_{\rho\alpha} - \partial_\alpha g_{\rho\sigma})$, $\mu = 0, 1, 2, 3$.

- Normalisation $g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = \epsilon$,
  $\epsilon = 0$ for light, $\epsilon = 1$ for particles.

Constants of motion

- Symmetries: spherical, axial
- Energy E, angular momentum L, Carter constant C.
- Reduction to a decoupled system of ODEs

In considered space-times

- Equations of motion given by (radial and latitudinal motion)

$$\left(x^i \frac{dx}{dy}\right)^2 = P(x;p).$$

  Here:

- P polynomial in x,
- $p = \{p_1, \ldots, p_n\}$ set of parameters of the space-time and the test particle (e.g. mass, energy, ...).

Neccessary condition

x, p such that $P(x;p) \geq 0$.
$\rightarrow$ Determination and classification of all possible orbit types.

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Find solution $x(y)$

$$\left(x^i \frac{dx}{dy} \right)^2 = P(x;p) \quad \Leftrightarrow \quad \int_{x_0}^{x} \frac{x^i dx}{\sqrt{P(x;p)}} = y - y_0 \,.$$

- Spherically symmetric space-times: $x(y) = r(\varphi)$.
- Axially symmetric space-times: $x(y) = r(\lambda)$ or $x(y) = \theta(\lambda)$, λ affine parameter (Mino time).

Periodicity

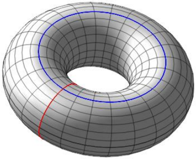
- $x(y)$ should be independent from the chosen path of integration
- If $\omega := \oint \frac{x^i dx}{\sqrt{P(x)}} \neq 0$ then: $x(y) = x(y - \omega)$.

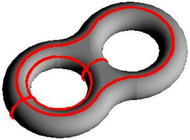
ODE as algebraic curve

With $w = \frac{dx}{dy}$: ODE is algebraic curve $x^{2i}w^2 - P(x;p) = 0$

- P of order 3 or 4, $i = 0$: elliptic curve, genus 1
- P of order $2g+1$, $2g+2$, $i < g$: hyperelliptic curve, genus g

Topology: Riemann surface





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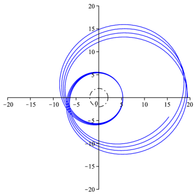
Topology: Riemann surface

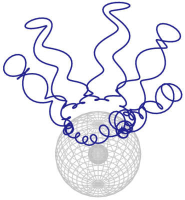
- The genus g corresponds to the number of 'holes' in the Riemann surface
- There are $2g$ independent closed integration pathes whose integrals do not vanish.

$\longrightarrow$ The solution function $x(y)$ needs to have $2g$ periods.

Elliptic curves

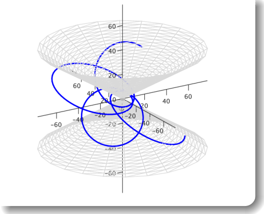
- Curve $w^2 = 4x^3 - g_2 x - g_3$: parametrisied by Weierstrass elliptic function: $x = \wp(z)$, $w = \wp'(z)$
- In Schwarzschild: $r(\varphi) = \frac{6M}{12\wp(\varphi - c) + 1}$, $c = c(r_0, \varphi_0)$
- Analogously: Kerr(-Newman), Taub-NUT, ... all Plebański-Demiański space-times with vanishing acceleration and cosmological constant

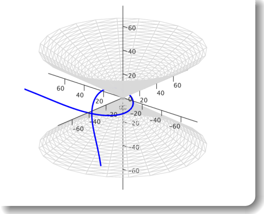




Hyperelliptic curves

- Functions in a single complex variable: no more than 2 periods
- Solution: more variables, restriction to one-dimensional submanifold
- Schwarzschild-de Sitter: $r(\varphi) = -M\frac{\sigma_1}{\sigma_2}(f(\varphi), \varphi)$, where $\sigma(f(x), x) = 0$ (theta divisor)
- Analogously: Kerr-de Sitter, Plebański-Demiański with $\Lambda \neq 0$





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Post-Newton and Schwarzschild

- Post-Newtonian: $\dot{\omega} \approx \frac{6\pi M}{p}$
- General formulation: $\dot{\omega} = 2\pi(\frac{\Omega_\varphi}{\Omega_r} - 1)$
- Analytic sol. Schwarzschild: $\dot{\omega} = \frac{4LK(k)}{\sqrt{(E^2-1)r_{\mathrm{p}}(r_{\mathrm{a}}-r_0))}} - 2\pi.$
- Analogously in Kerr; Lense Thirring effect: $2\pi(\frac{\Omega_\varphi}{\Omega_\theta} - 1)$

With cosmological constant

- Complete hyperelliptic integral: $K_{AB}(\vec{k}) := \int_0^1 \frac{(At+B)dt}{\sqrt{t\prod_{i=1}^3(1-k_i^2 t)}}$
- SdS: $\dot{\omega} = \frac{2c_0 K_{AB}(\vec{k})}{\sqrt{D}} + \sum_{i=1}^3 \frac{2c_i \Pi_{AB}(N_i,\vec{k})}{\sqrt{D}} - 2\pi$
- Computation: theta-constants, AGM for genus 2

Linear effects

- Schwarzschild ($M \neq 0$): Periapsis shift only effect
- Taub-NUT space-time ($M \neq 0, n \neq 0$): Motion on a cone instead of plane
- Kerr space-time ($M \neq 0, a \neq 0$): Periapsis shift changed, Lense-Thirring nonzero $\rightarrow$ precession of orbital plane in weak field. Both independent from direction of rotation of particle
- Kerr-Taub-NUT space-time ($M \neq 0, a \neq 0, n \neq 0$): precession of the orbital cone
- Kerr-Taub-NUT-de Sitter ($M \neq 0, a \neq 0, n \neq 0, \Lambda \neq 0$): Periaspsis shift is changed by Λ, but tiny.

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Geodesics

- Analytic pulsar timing for black hole companions
- Bending of light in all Plebański-Demiański space-times
- Geodesics in multipole space-times
- Geodesics in the higher-dimensional Meyers-Perry space-times

Algebro-geometric methods

- More general curves, e.g. quartic: $(w - P(x))^2 = Q(x)$

$\rightarrow$ Horava-Lifshitz, Gauss-Bonnet gravity

- Fast computation of Periods/Observables for higher genera
- Analytical solutions in fast semi-analytical codes