Novel geometric methods for quasi-local mass using isometric embeddings and curvature invariants

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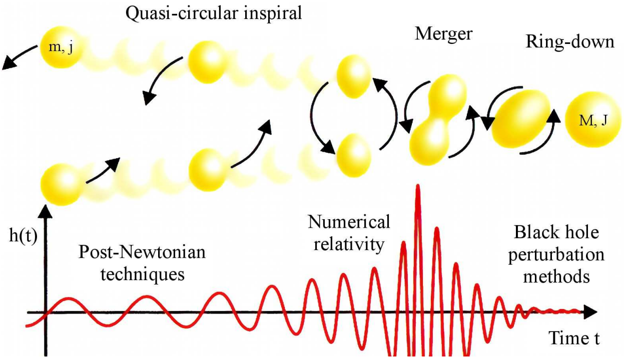
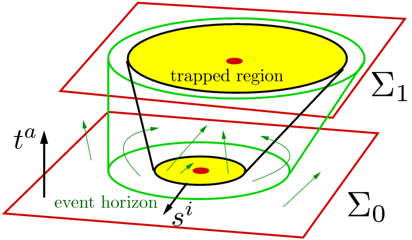
Motivation

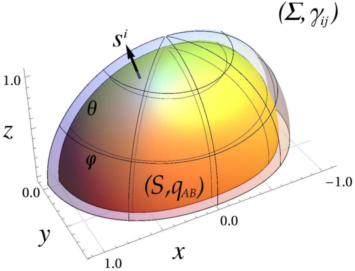
Diagnosing numerical relativistic two-body simulation

- Follow evolution of mass and spin during inspiral
- Measure initial and final masses
- Extract physical parameters for matching post-Newtonian and NumRel pieces to full gravitational waveforms



Measurement surface: 2-sphere bounding a region





Marginally-Outer-Trapped-Surface

$$\theta_{t+s} = ({}^qK_{ij} - {}^\gamma K_{ij})q^{ij} = 0$$
$$\theta_{t-s} < 0$$
$$m^2 = A\left[1 + (8\pi J/A)^2\right]/16\pi$$

Any 2-sphere S and $X_0 : S \to \mathbf{R}^3$

$$q_{ij} = \partial_i X_0^k \partial_j X_0^l \delta_{kl}$$

$m \sim \oint_S {}^qK^0 - Q dA$ with

$Q = {}^qK$ or $\sqrt{({}^qK)^2 - ({}^\gamma K_{ij} q^{ij})^2}$

Brown-York-93 or Liu-Yau-03

Measuring mass on MOTS: Two common approx.

MOTS is instantaneously Kerr

- Compute (m, J) by (A, L_{equator})
  $m = L/(4\pi)$
  $J^2 = (\frac{A}{8\pi})^2(m^2 16\pi/A - 1)$

  Smarr-73
- Or extrema of scalar curvature $(A, \mathcal{R}^{\min})$ Lovelace-etal-08

Problematic:

- L_{equator} not well-defined in general
- Several extrema may exist
- Single points represent geometry of the surface

MOTS is instantaneously axial IH

Solve $\mathcal{L}_\chi q_{ij} \approx 0$, $J[\chi] \sim \oint {}^\gamma K_{ij} \chi^i s^j dA$

1. Killing transport Dreyer-etal-03
2. Minimize functional Matzner-68
   $F[\chi] := < \mathcal{L}_\chi q_{ij} \mathcal{L}_\chi q^{ij} > / ||\chi||$
   Similar functional Cook-Whiting-07
   Solve EuLag eq Beetle-08

Problematic:

- (1.) χ^i path-dependent if q_{ij} non-axial, no divergence free
- (2.) No unique functional
- (2.) Solve PDE num. involved
- (1.), (2.) Normalisation of χ^i ambiguous!

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Practical alternative: Compute geometric invariants μ_n

$$\mu_n\left({}^q\hat{\mathcal{R}}\right) := \left\langle\left(\left\langle {}^q\hat{\mathcal{R}}\right\rangle - {}^q\hat{\mathcal{R}}\right)^n\right\rangle, \quad \langle\bullet\rangle := \frac{1}{A}\oint_S \bullet\, dA\,, \; < {}^q\hat{\mathcal{R}} >= 1$$

MOTS is instantaneously Kerr

Compute (m, J) by $(A, \mu_2({}^q\hat{\mathcal{R}}))$

$m^2 = \frac{A}{16\pi}(1 + \hat{c}^2)$
$J \;\; = \frac{A}{8\pi}\hat{c}$

$\mu_2({}^q\hat{\mathcal{R}}) = \frac{-15-70\hat{c}^2+\ldots}{80(1+\hat{c}^2)} + \frac{3(1+\hat{c}^2)^4}{16}\frac{\mathrm{atan}(\hat{c})}{\hat{c}}$

Advantage:

- Accounting for all points on S
- Averaging lowers the total error
- Technically practical: only surface integrals

MOTS is instantaneously axial IH

Solve algebraic system relating
$(\mu_n({}^q\hat{\mathcal{R}}), \mu_n(\mathrm{Im}\hat{\Psi}_2)) \leftrightarrow (\hat{L}_l, \hat{I}_l)$
axial multipole moments of ${}^q\hat{\mathcal{R}}, \mathrm{Im}\hat{\Psi}_2$
wrt Legendre P_l

$J \sim A \cdot \hat{L}_1$ Ashtekar-etal-04

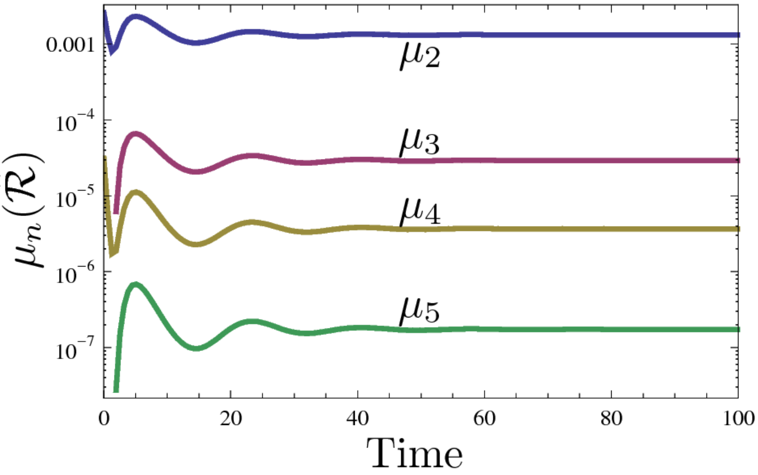
Advantage:

- No solution of PDE required (no additional num. errors)
- Unique multipolar approx. to perturbed axial metric

Applications to numerical simulations

Horizon of collapsing star: Saijo-11

Common horizon of 3D BBH simulation settling to Kerr: Jasiulek-09



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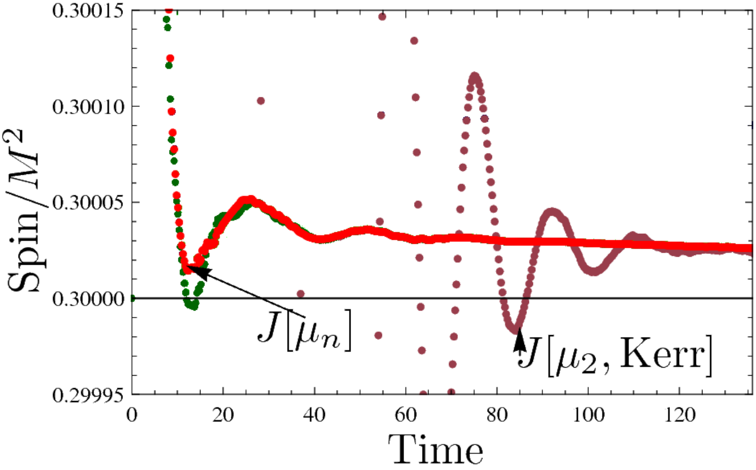
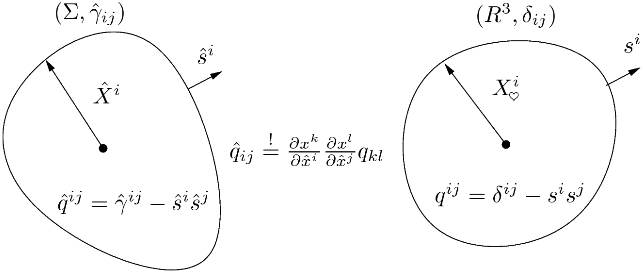


Weyl problem

Find an embedding X_0^i for 2-metric q_{AB} with ${}^q\mathcal{R} > 0$, $\det q_{AB} > 0$ such that $q_{AB} = \partial_A X_0^i \partial_B X_0^j \, \delta_{ij}$ (no "standard" num. method applicable)

Isometric embeddings provide

- Unique reference surface in $\mathbf{R}^3$ Nirenberg-53
- Quasi-local mass Brown-York-93 Liu-Yau-93
- Invariant coarse-graining of tensors in cosmology Korzynski-10



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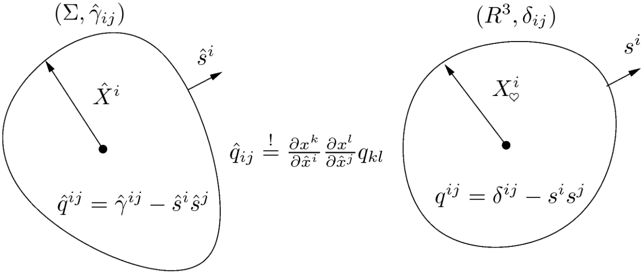
Attempts to solve Weyl problem:

- Constructing polyhedra in $\mathbf{R}^3$ Nollert-Herold-96
- Minim. functional Bondarescu-02 $F[X^i] := ||q_{AB} - \tilde{q}(X^i)_{AB}||$

No solution guaranteed!

- Multiple non-regular polyhedra $\exists$
- Minimisation problematic
- Do not address coordinate issue $x^i \leftrightarrow \hat{x}^i$



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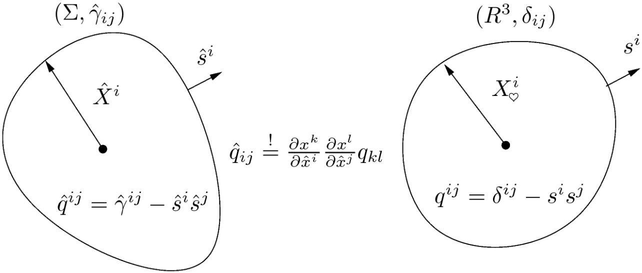
$\Rightarrow$ No practical application in NumRel exists !



Novel algorithm based on Weyl's method of continuity

Jasiulek-Korzynski-12

- Guaranteed to find embedding: Nirenberg-53
- Decomposes Weyl problem into linear elliptic 1D PDEs, ODEs
- Allows for implementation with spectral methods: predictable and reasonable computational cost
- Compatible with common 3+1 codes (arbitrary coords)

Consists of 3 main steps

1. Ricci flow $q(\hat{x})_{ij} \rightarrow {}^{\odot}q(\hat{x})_{ij}$ to round metric $\rightarrow$ conformal fac. σ
2. Determine canonical coords x^i such that ${}^{\odot}q(x)^{ij} = \delta^{ij} - x^i x^j / r^2$
3. Iterate linearised embedding flow $X^i_{\odot} \xrightarrow{\sigma \rightarrow 0} X^i_0$

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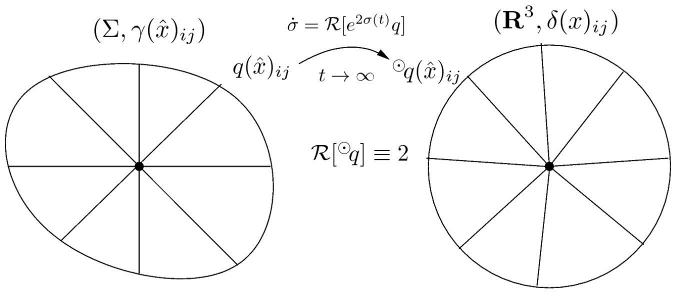
1. Step: Ricci flow: $q(\hat{x})_{ij} \to {}^{\odot}q(\hat{x})_{ij}$



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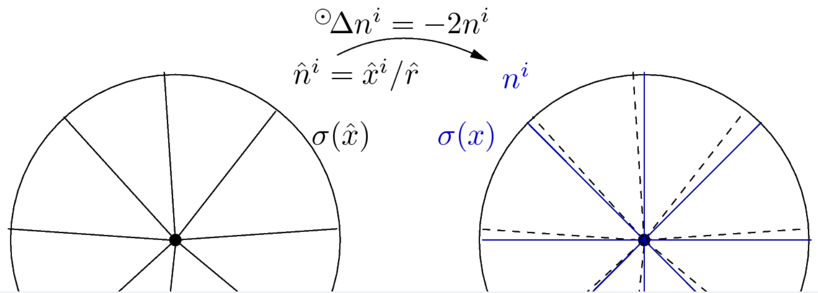
2. Step: Determine canonical coords x^i, $n^i = x^i / r$



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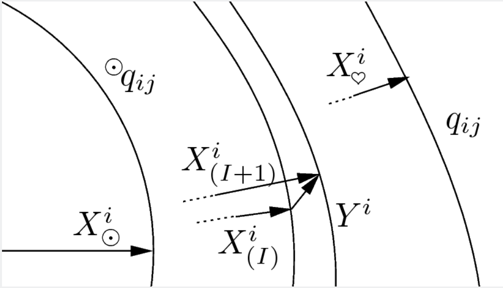
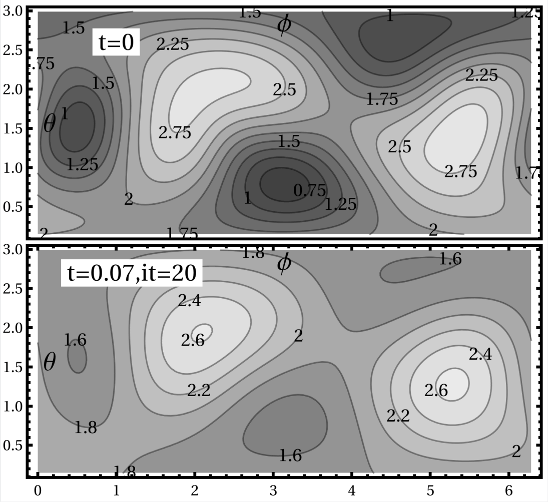
3. Iterate linearised embedding flow $X^i_{\odot} \xrightarrow{\sigma \to 0} X^i_0$

3. Step: $X^i_{\odot} \xrightarrow{\sigma \to 0} \left(X^i_{(I+1)} = X^i_{(I)} + Y^i \right) \xrightarrow{\sigma \to 0} X^i_0$

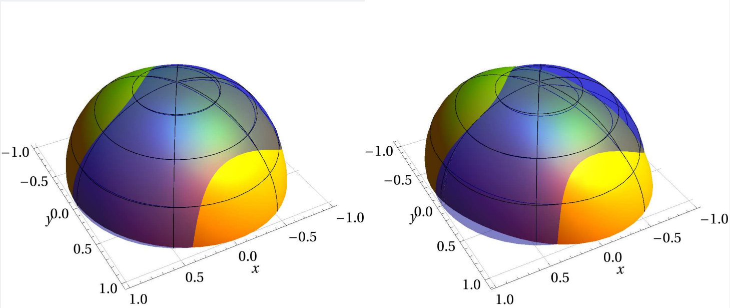


Test case: Cigar-shaped test metric + random piece

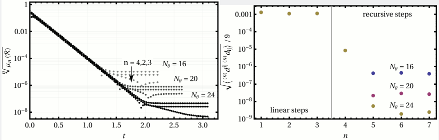
Ricci flow diffuses curvature gradients



Embedding flow at $t = 0.3$ and $t = 1$ with round sphere (blue)



Convergence



Summary

New methods for quasi-local mass on MOTS based on curvature invariants

- Practical computation of Kerr (m, J) through surface integrals (A, μ_2)
- Practical computation of axial (m, J) through surface integrals $(A, \mu_n({}^q\hat{\mathcal{R}}), \mu_n(\mathrm{Im}\hat{\Psi}_2))$
- No solution of PDE required, no construction of KVF χ^i
- Applicable to perturbed axial 2-metrics

New algorithm to compute isometric embeddings

- Solution for admissible 2-metrics guaranteed
- Decomposition into sub-steps, solvable via standard numerical methods
- Allows for implementation through spectral methods
- Compatible with existing 3+1 codes in NumRel
- Applications: Brown-York / Liu-Yau masses, numerical cosmology
- Methods solving Ricci flow, ${}^{\odot}\Delta$-EV problem interesting for: MOTS eq, Jang's eq, Minkowski problem

Thank you!

Numerical and technical details

Quasi-local mass through μ_n Jasiulek-09

- Representation of surface functions through spherical harmonics $h = \sum_{lm}^{l_{\max}} {}^{\Phi}[h]^{lm}\,\Phi^{lm}$, $\Phi^{lm} := (n^i\,\mathcal{N}_i^{lm})^l$
- Exact numerical integration up to $l_{\max} = N_\theta/2 - 1$ Discroll-Healy-94
- External Cartesian tensor basis for surface tensors and derivatives: e.g. $q_{ij} = \delta_{ij} - s_i s_j$, ${}^{q}D_i\chi_j = ({}^{\gamma}D_i\chi_j)^{\parallel} - {}^{q}K_{ij}\chi_s$
- Scalar curvature through Gauß theorem: ${}^{q}\mathcal{R} = {}^{\gamma}\mathcal{R} - 2\,{}^{\gamma}R_{ij}s^i s^j + {}^{q}\mathcal{K}^2 - {}^{q}K_{ij}\,{}^{q}K^{ij}$
- Implicit equations, algebraic system solved with `Mathematica`
- BBH simulation: `Cactus` toolkit, `Carpet` AMR driver, AEI-thorns now included in `Einstein Toolkit`, `AHFinderDirect`

Numerical and technical details

Algorithm for isometric embedding Jasiulek-Korzynski-12

- Representation of surface functions / tensor components through Y^{lm}
- Exact numerical integration up to Gauß-Legendre $l_{\max} = 2N_\theta - 1$
- External Cartesian tensor basis for surface tensors and derivatives: e.g. $q_{ij} = \delta_{ij} - s_i s_j$, ${}^qD_i\chi_j = ({}^\gamma D_i\chi_j)^\parallel - {}^qK_{ij}\chi_s$
- Axillary operations for surface functions / tensors:
  - Matrix inversion globally
  - Anti-differentiation (solve ODE system with spherical boundary)
  - Coordinate inversion $n^i(\hat{n}) \to \hat{n}^i(n)$
- Parabolic relaxation flow for elliptic PDEs

Linearised embedding equation

Instead of solving the full embedding equation linearise it around a known embedding X^i, $\hat{X}^i = X^i + Y^i$ for a small shift Y^i and $\hat{q}_{ij} = q_{ij} + d_{ij}$.

Linearise embedding equation

$$\hat{q}_{ij} = \partial_i \hat{X}^k \partial_j \hat{X}^l \delta_{kl} \quad \rightarrow \quad d_{ij} = 2\partial_{(i} Y^k \partial_{j)} X^l \delta_{kl} + \mathcal{O}((Y^i)^2)$$

Weyl's variable transformation $Y^i \rightarrow w, u_j$ with $w := -\epsilon^{ij}{}_k s^k D_i{}^{\parallel} Y_j$, $u_j := s_k(\partial_j Y^k)$ turns LEE into a second order elliptic PDE for w and ODEs for u^i.

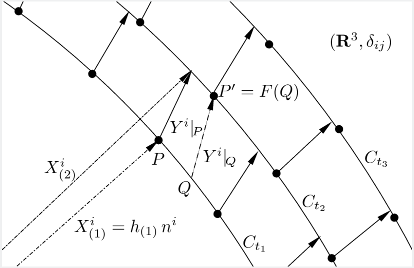
Variable transformed LEE

$${}^{\gamma}D_i \left[({}^{q}K^{(-1)})^{ij} \partial_j w \right] + {}^{q}\mathcal{K} w = T(d, Dd, K, DK)$$
$$\partial_j Y^i = \frac{1}{2} \left(w\, \epsilon^i{}_{jk} s^k + d^i{}_j \right) + s^i u_j$$

The elliptic operator $\mathcal{L}[\bullet]$ on l.h.s. of LEE has positive spectrum (except $l = 1$): solvable via parabolic relaxation flow $\dot{u} = \mathcal{L}[u] - T$.

Drifting of grid points during embedding flow

In general, the vector Y^i shifts grid points off the canonical grid which, thus, has to be determined implicitly on the next surface $X^i_{(I+1)}$.



Moreover, the target metric (thus $^{\odot}q, \sigma$) has to be transported under the mapping $X^i_{(I+1)} = X^i_{(I)} + Y^i$ from one surface to the next.

$$q_{ij}|_{P'} = \frac{\partial X^k_{(I)}}{\partial X^i_{(I+1)}} \frac{\partial X^l_{(I)}}{\partial X^j_{(I+1)}}\, q_{kl}|_Q$$