Regular and Chaotic Motion in General Relativity: The Case of a Massive Magnetic Dipole

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Outline

A massive magnetic dipole: Bonnor's solution

- Using standard coordinates (t, r, θ, ϕ) and geometrized units $c = G = 1$ the line element of Bonnor's exact solution (Bonnor, 1966) describing the static axisymmetric spacetime around massive magnetic dipole and corresponding vector potential A_ν are given as follows

$$\begin{aligned} ds^2 &= -\left(\frac{P}{Y}\right)^2 dt^2 + \frac{P^2 Y^2}{Q^3 Z}(dr^2 + Z d\theta^2) + \frac{Y^2 Z \sin^2\theta}{P^2} d\phi^2 \\ A_\nu &= \left(0, 0, 0, \frac{2abr\sin^2\theta}{P}\right), \\ \text{where } P &= r^2 - 2ar - b^2\cos^2\theta, \; Q = (r-a)^2 - (a^2+b^2)\cos^2\theta, \\ Y &= r^2 - b^2\cos^2\theta, \; Z = r^2 - 2ar - b^2. \end{aligned}$$

- two independent parameters a, b
- mass $M = 2a$, magnetic dipole moment $\mu = 2ab$, nonzero quadrupole mass moment Q fixed by the values of parameters a and b: $Q = \frac{\mu}{M} - \frac{1}{4}M^3$

Properties of Bonnor's solution

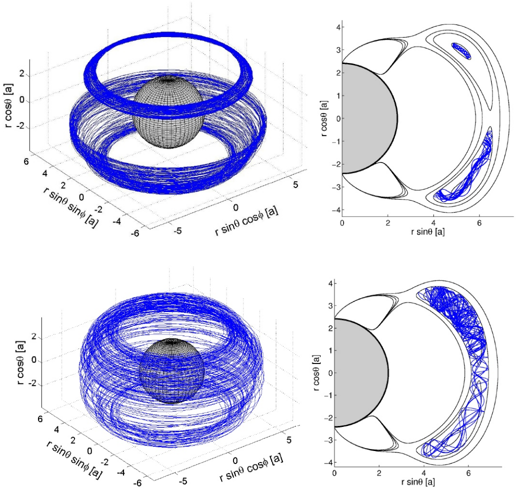
- It represents the **magnetostatic limit** of a more general exact solutions (Pachón et al., 2006; Manko et al. 2000) suggested to describe the **exterior field of a neutron star** (adding spin, electric charge, current octupole moment and quadrupole mass moment as extra free parameters).
- **Asymptotically flat**; with $a = 0$ the spacetime is exactly flat with its spatial part described by prolate spheroidal coordinates (r, θ, ϕ).
- With $b = 0$ the solution **does not reduce to the Schwarzschild metric** as it keeps quadrupole mass moment (no spherical symmetry). Non-magnetized Bonnor spacetime belongs to the class of solutions found by Darmois (1927).
- The resulting field may be interpreted as a **sum of two oppositely charged extremal Reissner-Nordström black holes** (with charge $\pm a$) placed on the symmetry axis at $\pm b$.
- The solution has relatively complicated singular behviour at $P = 0$, $Q = 0$, $Z = 0$ and $Y = 0$ (analyzed by Ward, 1975; Emparan, 2000). However, we are interested in the regular part of the spacetime only. Therefore we restrict ourselves to $Z > 0$ which translates to the condition $r > r_{\rm h} \equiv a + \sqrt{a^2 + b^2}$. **We investigate the test particle dynamics above the horizon $r_{\rm h}$ only**.

Equations of motion & effective potential

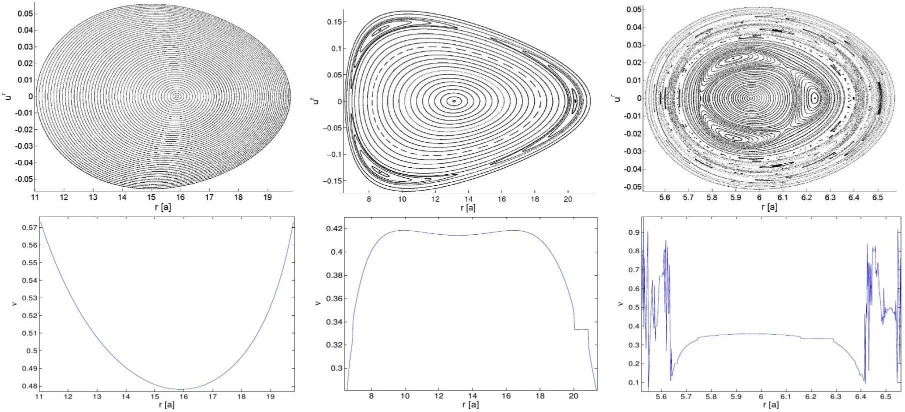
- Employing the super-Hamiltonian $\mathcal{H} = \frac{1}{2}g^{\mu\nu}(\pi_\mu - qA_\mu)(\pi_\nu - qA_\nu)$ the equations of motion are given as

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \equiv p^\mu = \frac{\partial \mathcal{H}}{\partial \pi_\mu}, \quad \frac{d\pi_\mu}{d\lambda} = -\frac{\partial \mathcal{H}}{\partial x^\mu},$$

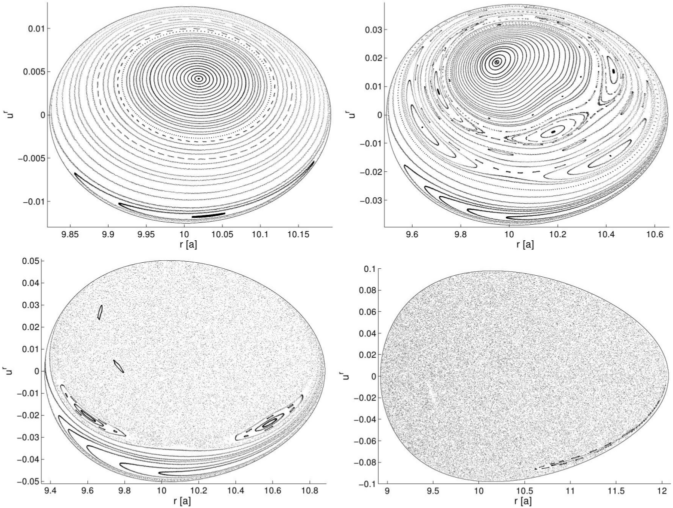
  where $\lambda = \tau/m$ is the affine parameter, τ denotes the proper time, and p^μ is the standard kinematical four-momentum for which the first equation reads $p^\mu = \pi^\mu - qA^\mu$.
- Two obvious **constants of motion**, the **energy** $E \equiv -\pi_t$ and **angular momentum** $L \equiv -\pi_\varphi$ which correspond to cyclic coordinates t, ϕ. **Hamiltonian is autonomous** $\mathcal{H} \neq \mathcal{H}(\lambda)$ – the motion generally occurs on the 3-dimensional hypersurface in the phase space. If there is yet another independent integral of motion analogically to the Carter's fourth constant $\mathcal{L}$ in the Kerr-Newman spacetime (Carter, 1968) the system is completely integrable with no traces of deterministic chaos (motion on 2-dim hypersurface $\Rightarrow$ curves on the Poincaré surface of section). On the other hand, if there is no extra integral, the system is non-integrable which means that regular and chaotic orbits coexist in its phase space (presence of area-filling orbits on the surface of section).
- We construct **two-dimensional effective potential** $V_{\mathrm{eff}}(r, \theta;\ a,\ b,\ L)$ in order to locate off-equatorial potential minima. We find that **off-equatorial stable orbits are allowed** in this setup. We also locate off-equatorial orbits of neutral test particles ($q = 0$) which clearly manifests profound difference between Bonnor's exact solution and test field solutions which only act on the ionised test particles as they do not affect the geometry of the spacetime itself.



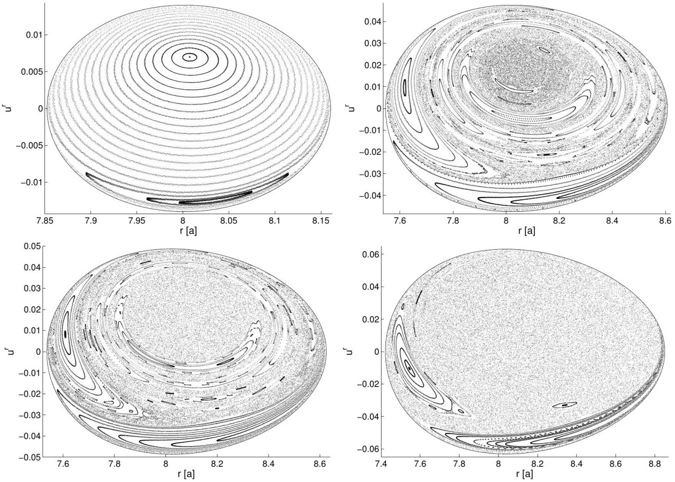
Off-equatorial and cross-equatorial trajectories of charged test particle ($L = -2.356\ a$, $q = 5.581$) in the Bonnor spacetime ($b = 1\ a$). In the upper left panel we present a stereometric projection of two trajectories: the upper one with $E = 0.8169$ shows ordered motion while with the higher energy the dynamics acquires partially chaotic properties (bottom trajectory with $E = 0.8182$). We plot the poloidal (r, θ) projection of these trajectories along with several iso-contours of the effective potential in the upper right panel. Increasing the energy to the value $E = 0.819$ the trajectory is allowed to cross the equatorial plane freely and is now of the fully chaotic nature (bottom panels). All particles were launched at $r(0) = 6$, $\theta(0) = \pi/3, 2\pi/3$, respectively, with $u^r(0) = 0$. Grey color marks $r = r_h$ surface in all plots.



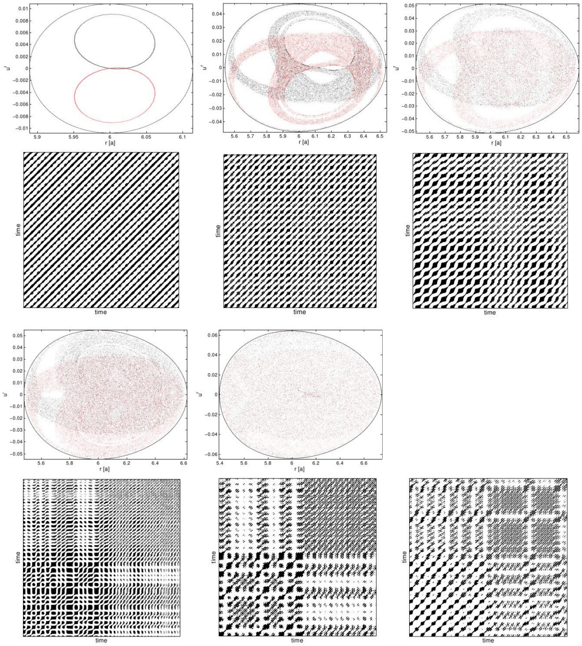
Comparison of dynamics in equatorial potential wells ($\theta_{\rm min} = \theta_{\rm sec} = \pi/2$). Common parameters of the trajectories in the left panels are $E = 0.95$, $L = 7.21\ a$, $q = 0$ and $b = 0$. Both the surface of section as well as behaviour of rotation number ν strongly suggests that system is integrable in this case. Middle panels show the situation for particles with $E = 0.94$, $L = 6.11\ a$, $q = 0$ and $b = 2.85\ a$. The chain of Birkhoff islands develops here. Although we know that there are some integrable system with resonant islands of single multiplicity (Contopoulos, 2002) here its presence arouses suspicion of non-integrability since none were present for $b = 0$. Right panels are plotted for $E = 0.82$, $L = -2.73\ a$, $q = 4.72$ and $b = 1\ a$ which leads to the equal value of $r_{\rm min}/r_{\rm h}$ as in the previous uncharged case. Here the KAM curves of quasiperiodic orbits are present as well as several Birkhoff chains of islands corresponding to the resonances of intrinsic frequencies of the system. These are interwoven with pronounced chaotic layers. Such picture is typical for considerably perturbed system far from integrability.



Poincaré surfaces of section ($\theta_{\rm sec} = \pi/3$) of electrically neutral particles in the Bonnor spacetime ($b = 5.9771\ a$). Particles with $L = 3.6743\ a$ are launched from the vicinity of the off-equatorial potential well ($r_{min} = 10\ a$, $\theta_{\rm min} = \theta_{\rm sec} = \pi/3$ and $V_{\rm min} = 0.8717$) with various values of energy. Upper left panel shows the section for the level $E = 0.8718$ (small off-equatorial lobe), in the upper right we set $E = 0.873$ (large off-equatorial lobe), $E = 0.8739$ produces cross-equatorial lobe which just emerged from symmetric off-equatorial lobes (bottom left panel) while with $E = 0.88$ we obtain large cross-equatorial lobe which almost opens.



Poincaré surfaces of section ($\theta_{\rm sec} = \pi/3$) of electrically charged particles ($q = 0.1259$) in the Bonnor spacetime ($b = 4.5393\,a$). Particles with $L = -3.5486\,a$ are launched from the vicinity of the off-equatorial potential well ($r_{min} = 8\,a$, $\theta_{\rm min} = \theta_{\rm sec} = \pi/3$ and $V_{\rm min} = 0.8475$) with various values of energy. Upper left panel shows the section for the level $E = 0.8477$ (small off-equatorial lobe), in the upper right we set $E = 0.8495$ (large off-equatorial lobe), $E = 0.8496$ produces cross-equatorial lobe which just emerged from symmetric off-equatorial lobes (bottom left panel) while with $E = 0.851$ we obtain large cross-equatorial lobe.



RECURRENCE PLOTS

- simple construction regardless the dimension of the phase space (unlike Poincaré surfaces)
- regular orbits lead to the simple diagonal pattern while the deterministic chaos manifests itself by the complex structure in the recurrence plot
- may be quantified easily (Recurrence Quantification Analysis)
- review on recurrence analysis: Marwan (2006)

Summary

- From the numerical study presented above we conclude that **geodesic motion in a non-magnetized Bonnor's spacetime ($b = 0$) is highly probably integrable**. Claim of integrability could be further supported by (non-conclusive) Painlevé test of integrability. However, it can be fully proved only by finding extra integral of motion analytically.
- **Magnetic parameter** b **introduces non-integrable perturbation**. **Charge** of the particle acts as an **extra perturbation** which shifts magnetized system even **farther from the integrability**.
- We studied the **effect of particle's energy** E on the degree of chaos found in the system concluding that it acts as a **trigger for chaotic motion**. When the energy is gradually increased the system undergoes continuous transition from ordered motion to chaotic dynamics being almost fully ergodic on the given hypersurface. We illustrated such transition by means of Poincaré surfaces of section and recurrence plots constructed from the trajectory of particular ionised particle.
- **Unlike previously analyzed test field solutions** (see Kopáček et al., 2010; Kovář et al., 2010, 2008) which acted as a perturbation to the integrable system of Kerr or Schwarzschild black holes here the **off-equatorial orbits** are found also for **neutral particles**.