Relativity and Gravitation – 100 Years after Einstein in Prague
June 27, 2012

Can effects of **Quantum Gravity** be observed in the **Cosmic Microwave Background**?

based on: – Claus Kiefer and M. K., Phys. Rev. Lett. **108**, 021301 (2012).
– _____ , Int. J. Mod. Phys. D **21**, 1241001 (2012), arXiv:1205.5161.
(First Award in the 2012 Essay Contest of the Gravity Research Foundation)

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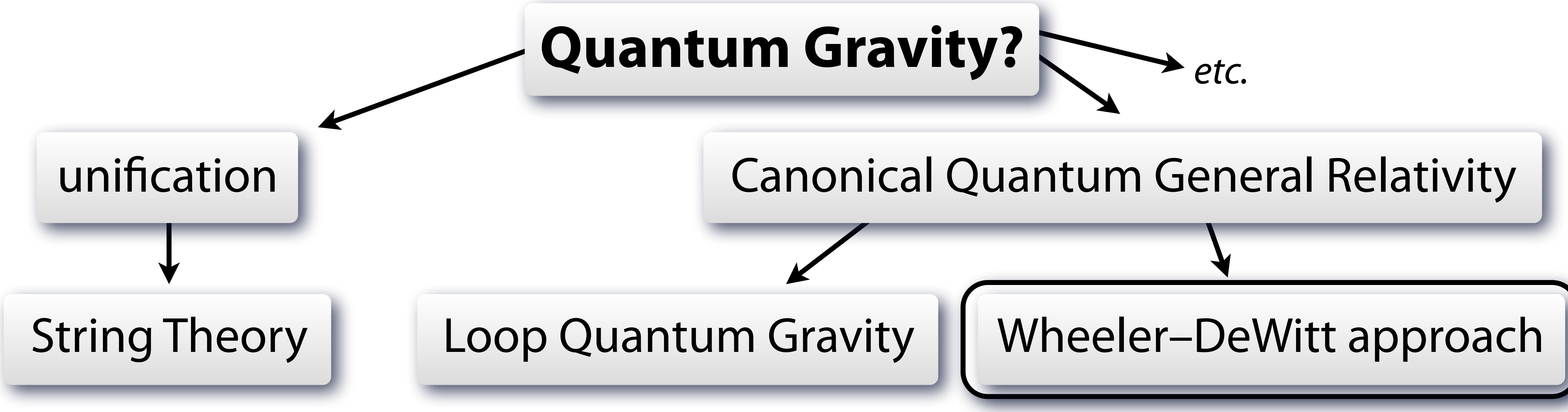
The problem with Quantum Gravity



- observational guidance needed to distinguish the candidate theories
- *problem:* quantum-gravitational effects might only become dominant in the Planck regime

$$m_{\rm P} = \sqrt{\frac{\hbar c}{G}} \simeq 1.22 \times 10^{19}\,{\rm GeV}/c^2$$

- effects are expected for: ➔ black holes *(Hawking radiation)*
  ➔ very early universe *(Cosmic Microwave Background)*

Wheeler–DeWitt approach *(Quantum Geometrodynamics)*

- canonical quantization of Hamiltonian formulation of General Relativity
- 3+1 decomposition by foliating spacetime (ADM formalism)
- resulting equation: Wheeler–DeWitt equation

$$\hat{\mathcal{H}}\,\Psi[h_{ij}(\mathbf{x}),\phi(\mathbf{x})] = 0$$

wave functional ↗ 3-metric ↖ matter field ↖

- functional differential equation on "superspace" ← space of all 3-geometries
- timeless (GR: *dynamical time* **vs.** QM: *absolute time* ➔ QG: *no time*)
- Born–Oppenheimer approximation with respect to $m_{\mathrm{P}}^2 \propto G^{-1}$
  ➔ Hamilton–Jacobi equation of GR ➔ recovery of Einstein eq.
  ➔ functional Schrödinger eq. for matter field in curved spacetime; WKB time
  ➔ quantum-gravitational correction terms to Schrödinger eq. $\propto m_{\mathrm{P}}^{-2}$

details: Kiefer and Singh, Phys. Rev. D **44**, 1067 (1991).

➡ *dominant QG contribution for the power spectrum of cosmol. perturbations?*

Quantum-cosmological model

- *simplest model:* inflationary universe with perturbations of a scalar field
- *background universe:* flat Friedmann–Lemaître universe with scale factor $a \equiv \exp(\alpha)$ and inflaton field ϕ
- slow roll: $\dot{\phi}^2 \ll |\mathcal{V}(\phi)|$ ➔ inflaton potential: $\mathcal{V}(\phi) = \frac{1}{2} m^2 \phi^2 \approx \text{const.}$
- set: $\hbar = c = 1$, define: $m_\mathrm{P} \equiv \sqrt{\frac{3\pi}{2G}}$, rescale: $\phi \to \frac{1}{\sqrt{2}\,\pi}\,\phi$

➔ Wheeler–DeWitt equation in <u>mini</u>superspace (α, ϕ):

$$\mathcal{H}_0 \Psi_0 = \frac{1}{2}\,\mathrm{e}^{-3\alpha} \left[\frac{1}{m_\mathrm{P}^2} \frac{\partial^2}{\partial \alpha^2} - \frac{\partial^2}{\partial \phi^2} + \mathrm{e}^{6\alpha}\, m^2 \phi^2 \right] \Psi_0(\alpha, \phi) = 0$$

- assume: $\frac{\partial^2 \Psi_0}{\partial \phi^2} \ll \mathrm{e}^{6\alpha}\, m^2 \phi^2\, \Psi_0$, substitute: $m\phi \to m_\mathrm{P} H$

$$\mathcal{H}_0 \Psi_0 = \frac{1}{2}\,\mathrm{e}^{-3\alpha} \left[\frac{1}{m_\mathrm{P}^2} \frac{\partial^2}{\partial \alpha^2} + \mathrm{e}^{6\alpha}\, m_\mathrm{P}^2 H^2 \right] \Psi_0(\alpha) = 0$$

Wheeler–DeWitt equation with perturbations

- add perturbations to the scalar field: $\phi \to \phi(t) + \delta\phi(\mathbf{x}, t)$
- decompose into Fourier modes: $\delta\phi(\mathbf{x}, t) = \sum_k f_k(t)\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}$

➡ WDW eq. with perturb.:
$$\left[\mathcal{H}_0 + \sum_{k=1}^{\infty} \mathcal{H}_k\right] \Psi\left(\alpha, \{f_k\}_{k=1}^{\infty}\right) = 0$$

with $\mathcal{H}_k = \frac{1}{2}\, \mathrm{e}^{-3\alpha} \left[-\frac{\partial^2}{\partial f_k^2} + \left(k^2\, \mathrm{e}^{4\alpha} + m^2\, \mathrm{e}^{6\alpha}\right) f_k^2 \right]$

similar to: Halliwell and Hawking, Phys. Rev. D **31**, 1777 (1985).

- product ansatz: $\Psi\left(\alpha, \{f_k\}_{k=1}^{\infty}\right) = \Psi_0(\alpha) \prod_{k=1}^{\infty} \widetilde{\Psi}_k(\alpha, f_k)$
- wave function for each mode $\Psi_k(\alpha, f_k) \equiv \Psi_0(\alpha) \widetilde{\Psi}_k(\alpha, f_k)$

$$\frac{1}{2}\, \mathrm{e}^{-3\alpha} \left[\frac{1}{m_\mathrm{P}^2} \frac{\partial^2}{\partial \alpha^2} + \mathrm{e}^{6\alpha}\, m_\mathrm{P}^2 H^2 - \frac{\partial^2}{\partial f_k^2} + W_k(\alpha) f_k^2 \right] \Psi_k(\alpha, f_k) = 0$$

$W_k(\alpha) \equiv k^2\, \mathrm{e}^{4\alpha} + m^2\, \mathrm{e}^{6\alpha}$

Semiclassical approximation

- Born–Oppenheimer approximation, WKB ansatz: $\Psi_k(\alpha, f_k) = \mathrm{e}^{\mathrm{i}\, S(\alpha, f_k)}$

- expansion of $S(\alpha, f_k)$: $S = m_\mathrm{P}^2\, S_0 + m_\mathrm{P}^0\, S_1 + m_\mathrm{P}^{-2}\, S_2 + \ldots$

- insert WKB ansatz into WDW eq. and equate terms of equal power of m_P

  - $\mathcal{O}(m_\mathrm{P}^2)$: Hamilton–Jacobi equation: $\left[\frac{\partial S_0}{\partial \alpha}\right]^2 - \mathrm{e}^{6\alpha} H^2 = 0$

  - $\mathcal{O}(m_\mathrm{P}^0)$: define $\psi_k^{(0)}(\alpha, f_k) \equiv \gamma(\alpha)\, \mathrm{e}^{\mathrm{i}\, S_1(\alpha, f_k)}$

    ➔ introduce WKB time:

    $$\frac{\partial}{\partial t} \equiv -\, \mathrm{e}^{-3\alpha} \frac{\partial S_0}{\partial \alpha} \frac{\partial}{\partial \alpha}$$

    ➔ **Schrödinger equation:**

    $$\mathrm{i}\, \frac{\partial}{\partial t} \psi_k^{(0)} = \mathcal{H}_k \psi_k^{(0)}$$

  - $\mathcal{O}(m_\mathrm{P}^{-2})$: **quantum-gravitationally corrected Schrödinger eq.:**

$$\mathrm{i}\, \frac{\partial}{\partial t} \psi_k^{(1)} = \mathcal{H}_k \psi_k^{(1)} - \frac{\mathrm{e}^{3\alpha}}{2\, m_\mathrm{P}^2\, \psi_k^{(0)}} \left[\frac{\left(\mathcal{H}_k\right)^2}{V} \psi_k^{(0)} + \mathrm{i}\, \frac{\partial}{\partial t} \left(\frac{\mathcal{H}_k}{V} \right) \psi_k^{(0)} \right] \psi_k^{(1)}$$

V ← $\mathrm{e}^{6\alpha} H^2$ → V

Solution to the <u>un</u>corrected Schrödinger equation

- use Gaussian ansatz: $\psi_k^{(0)}(t, f_k) = \mathcal{N}_k^{(0)}(t)\, \mathrm{e}^{-\frac{1}{2}\,\Omega_k^{(0)}(t)\, f_k^2}$

➡ set of differential equations:

$$\dot{\mathcal{N}}_k^{(0)}(t) = -\frac{\mathrm{i}}{2}\,\mathrm{e}^{-3\alpha}\,\mathcal{N}_k^{(0)}(t)\,\Omega_k^{(0)}(t)$$

$$\dot{\Omega}_k^{(0)}(t) = \mathrm{i}\,\mathrm{e}^{-3\alpha}\left[-\left(\Omega_k^{(0)}(t)\right)^2 + W_k(t)\right]$$

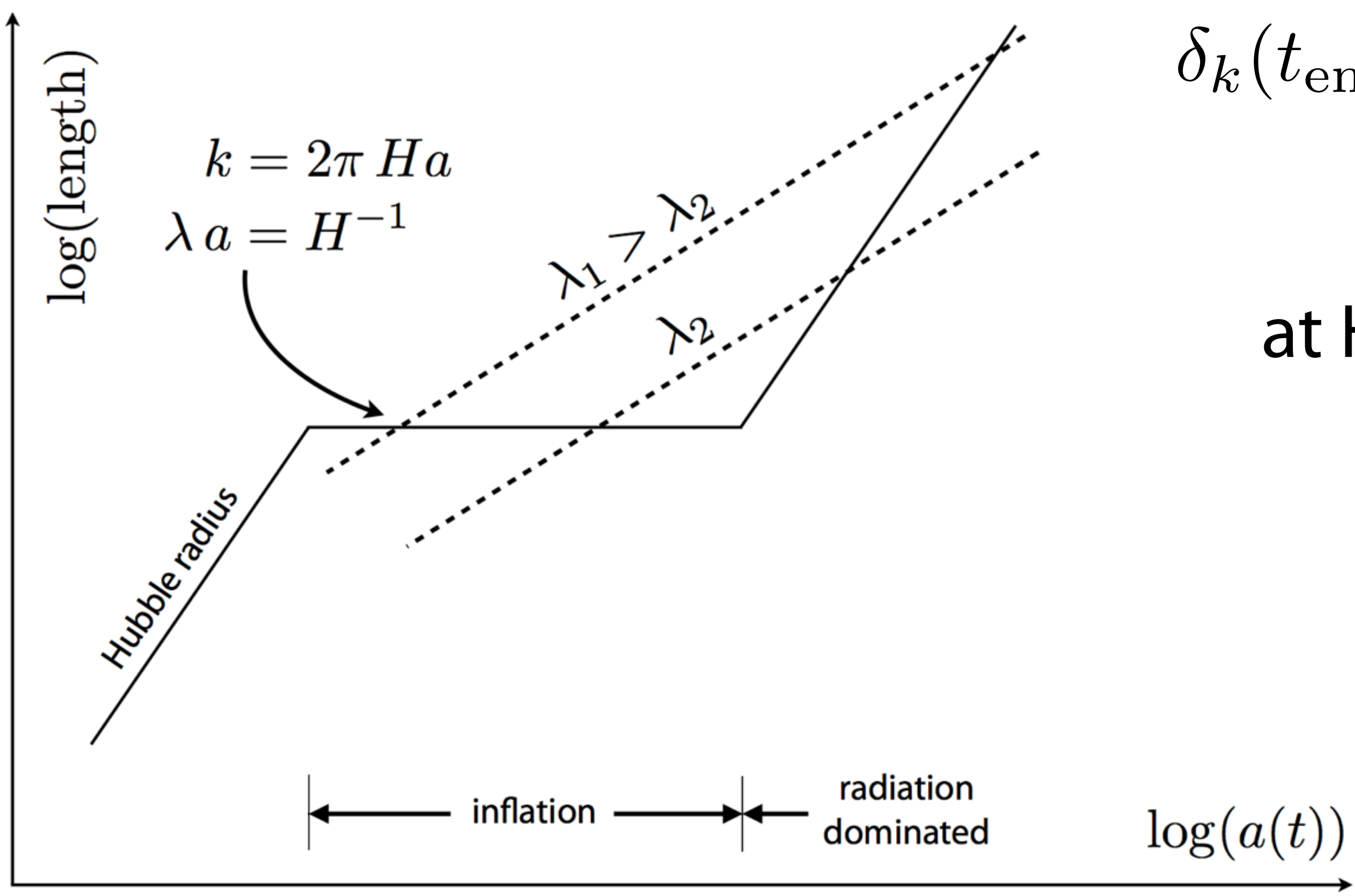
➡ solution: $\Omega_k^{(0)}(t) = \dfrac{k^2 a^2}{k^2 + H^2 a^2}\,(k + \mathrm{i}\,Ha) + \mathcal{O}\left(\dfrac{m^2}{H^2}\right)$

- use this to calculate the power spectrum of the scalar field perturbations
  - density contrast in slow-roll regime: $\delta_k(t) \approx \dfrac{\delta\rho_k(t)}{\mathcal{V}_0} = \dfrac{\dot{\phi}(t)\,\dot{\sigma}_k(t)}{\mathcal{V}_0}$
  - $\sigma_k(t)$: classical quantity related to quantum-mechan. quantity $f_k(t)$

$$\sigma_k^2(t) := \langle\psi_k|f_k^2|\psi_k\rangle = \sqrt{\frac{\mathfrak{Re}\,\Omega_k}{\pi}}\int_{-\infty}^{\infty} f_k^2\,\mathrm{e}^{-\frac{1}{2}[\Omega_k^*(t)+\Omega_k(t)]f_k^2}\,\mathrm{d}f_k = \frac{1}{2\,\mathfrak{Re}\,\Omega_k(t)}$$

Uncorrected power spectrum

- evaluate $\delta_k(t)$ at the time when mode reenters Hubble radius $t_{\rm enter}$
- relation between $t_{\rm enter}$ and the time the mode exits Hubble radius:



$$\delta_k(t_{\rm enter}) = \frac{4}{3}\frac{\mathcal{V}_0}{\dot\phi^2}\,\delta_k(t_{\rm exit}) = \frac{4}{3}\left.\frac{\dot\sigma_k(t)}{\dot\phi(t)}\right|_{t_{\rm exit}}$$

at Hubble radius exit: $k = 2\pi\, H a$

$$\Rightarrow \quad \left|\dot\sigma_k^{(0)}(t)\right|_{t_{\rm exit}} \propto \frac{H^2}{k^{\frac{3}{2}}}$$

➡ **power spectrum:** $\Delta^2_{(0)}(k) := 4\pi k^3\,|\delta_k(t_{\rm enter})|^2 \propto \dfrac{H^4}{\left|\dot\phi(t)\right|^2_{t_{\rm exit}}} \approx \text{const.}$

Solution to the QG corrected Schrödinger equation

$$\mathrm{i}\,\frac{\partial}{\partial t}\,\psi_k^{(1)} = \mathcal{H}_k\psi_k^{(1)} - \frac{\mathrm{e}^{3\alpha}}{2\,m_\mathrm{P}^2\,\psi_k^{(0)}}\left[\frac{\left(\mathcal{H}_k\right)^2}{V}\,\psi_k^{(0)} + \mathrm{i}\,\frac{\partial}{\partial t}\left(\frac{\mathcal{H}_k}{V}\right)\psi_k^{(0)}\right]\psi_k^{(1)}$$

- first correction term dominant ➔ unitarity-violating term negligible
- differential equation of fourth order ➔ approximation necessary
  ➔ modified Gaussian ansatz:

$$\psi_k^{(1)}(t, f_k) = \left(\mathcal{N}_k^{(0)}(t) + \frac{1}{m_\mathrm{P}^2}\,\mathcal{N}_k^{(1)}(t)\right)\exp\left[-\frac{1}{2}\left(\Omega_k^{(0)}(t) + \frac{1}{m_\mathrm{P}^2}\,\Omega_k^{(1)}(t)\right)f_k^2\right]$$

  - yields a set of differential equations

$$\dot{\Omega}_k^{(1)}(t) \approx -2\,\mathrm{i}\,\mathrm{e}^{-3\alpha}\,\Omega_k^{(0)}(t)\left(\Omega_k^{(1)}(t) - \frac{3}{4V(t)}\left[\left(\Omega_k^{(0)}(t)\right)^2 - W_k(t)\right]\right)$$

  - boundary condition: $\Omega_k^{(1)}(t) \to 0$ as $t \to \infty$
  - can be solved by the method of variation of constants

Quantum-gravitational correction term

- QG corr.: $\left|\dot{\sigma}_n^{(1)}(t)\right| = \left| \frac{1}{\sqrt{2}} \frac{\mathrm{d}}{\mathrm{d}t} \left[\left(\mathfrak{Re}\left[\Omega_k^{(0)}(t)\right] + \frac{1}{m_{\mathrm{P}}^2} \mathfrak{Re}\left[\Omega_k^{(1)}(t)\right] \right)^{-\frac{1}{2}} \right] \right|$

- correction can be incorporated into a correction term C_k:

$$\left|\dot{\sigma}_k^{(1)}\right|_{t_{\mathrm{exit}}} \simeq |C_k| \left|\dot{\sigma}_k^{(0)}\right|_{t_{\mathrm{exit}}}$$

- explicit form of C_k obtained by (numerical) integration:

$$C_k = \left(1 - \frac{43.56}{k^3} \frac{H^2}{m_{\mathrm{P}}^2}\right)^{-\frac{3}{2}} \left(1 - \frac{189.18}{k^3} \frac{H^2}{m_{\mathrm{P}}^2}\right)$$

➡ QG corrected power spectrum: $\Delta^2_{(1)}(k) = \Delta^2_{(0)}(k)\, C_k^2$

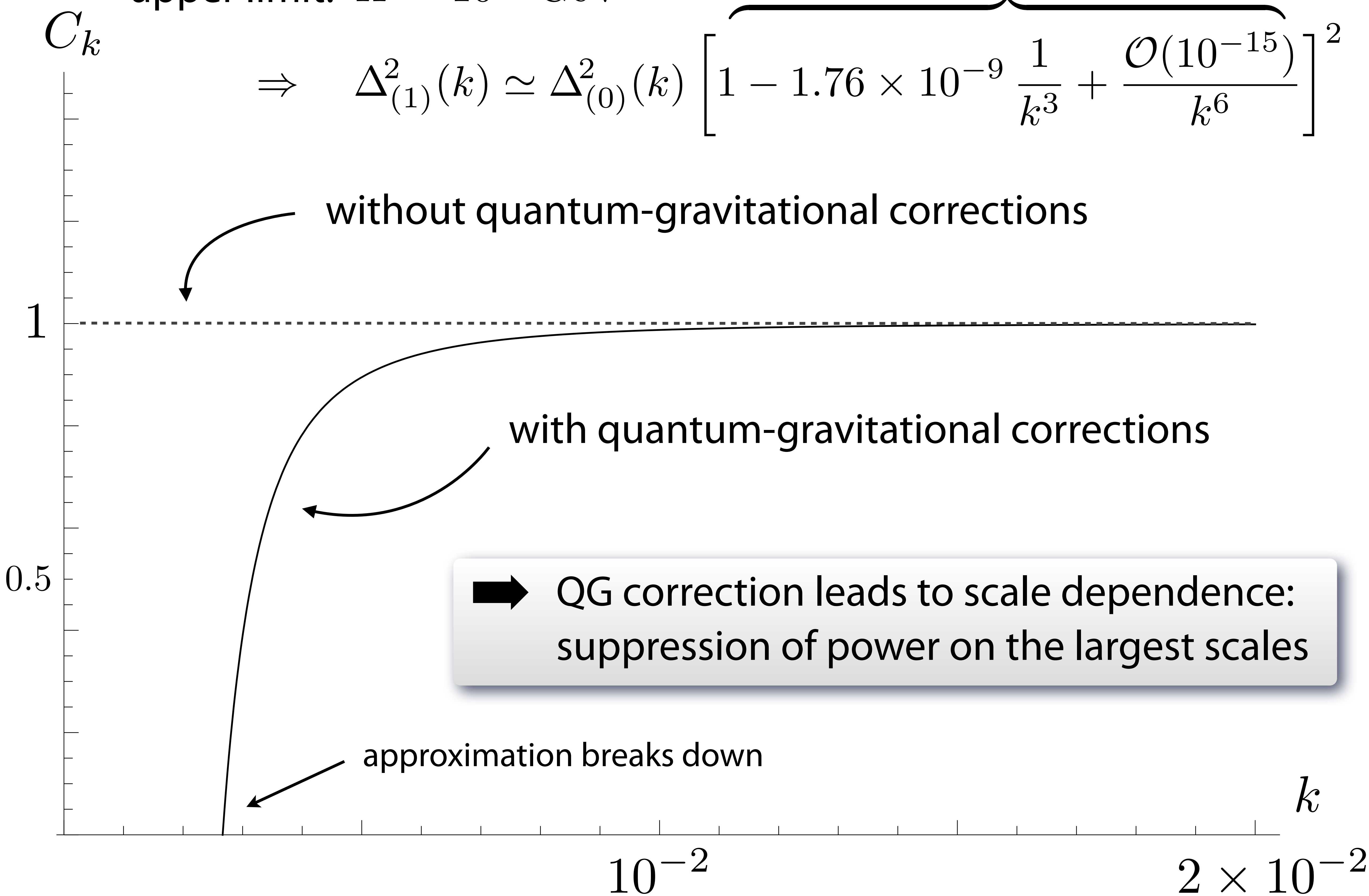
- Taylor expansion:

$$\Delta^2_{(1)}(k) \simeq \Delta^2_{(0)}(k) \left[1 - \frac{123.83}{k^3} \frac{H^2}{m_{\mathrm{P}}^2} + \frac{1}{k^6} \mathcal{O}\left(\frac{H^4}{m_{\mathrm{P}}^4} \right) \right]^2$$

Quantum-gravitationally corrected power spectrum

upper limit: $H \simeq 10^{14}\,\mathrm{GeV}$

$$\Rightarrow \quad \Delta^2_{(1)}(k) \simeq \Delta^2_{(0)}(k) \Big[\overbrace{1 - 1.76 \times 10^{-9}\,\frac{1}{k^3} + \frac{\mathcal{O}(10^{-15})}{k^6}}^{C_k}\Big]^2$$



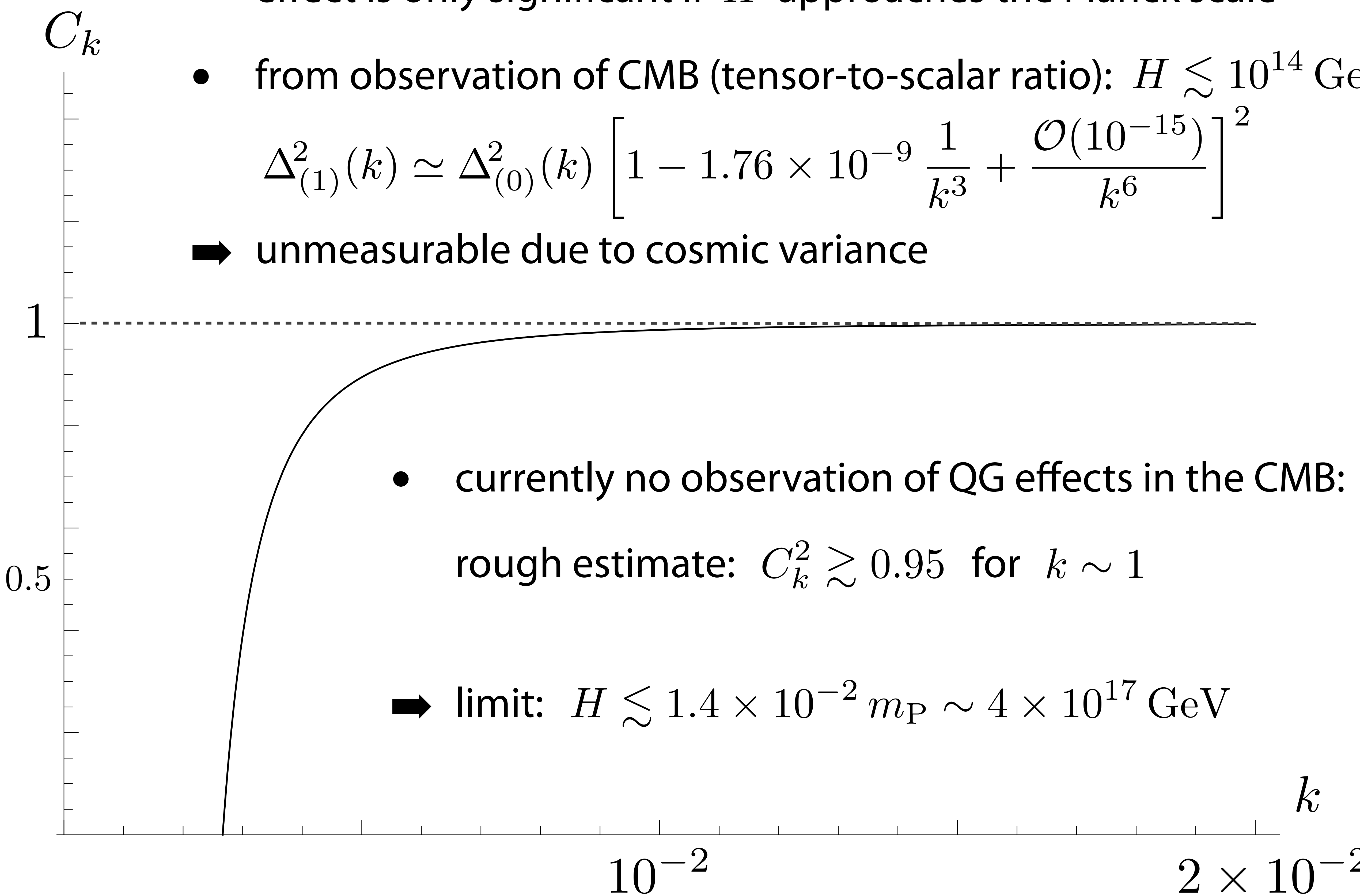
➡ QG correction leads to scale dependence: suppression of power on the largest scales

Measurability of the quantum-gravitational correction

- effect is only significant if H approaches the Planck scale
- from observation of CMB (tensor-to-scalar ratio): $H \lesssim 10^{14}\,\mathrm{GeV}$

$$\Delta^2_{(1)}(k) \simeq \Delta^2_{(0)}(k)\left[1 - 1.76\times 10^{-9}\,\frac{1}{k^3} + \frac{\mathcal{O}(10^{-15})}{k^6}\right]^2$$

➡ unmeasurable due to cosmic variance



- currently no observation of QG effects in the CMB:

rough estimate: $C_k^2 \gtrsim 0.95$ for $k \sim 1$

➡ limit: $H \lesssim 1.4 \times 10^{-2}\, m_{\mathrm{P}} \sim 4 \times 10^{17}\,\mathrm{GeV}$

Summary

- quantization of an inflationary universe with perturbations of a scalar field
- quantum-gravitational correction to the power spectrum of these perturbations
  - induces scale dependence, suppression of power on largest scales
  - out of experimental reach due to cosmic variance
  - weak limit on Hubble constant during inflation

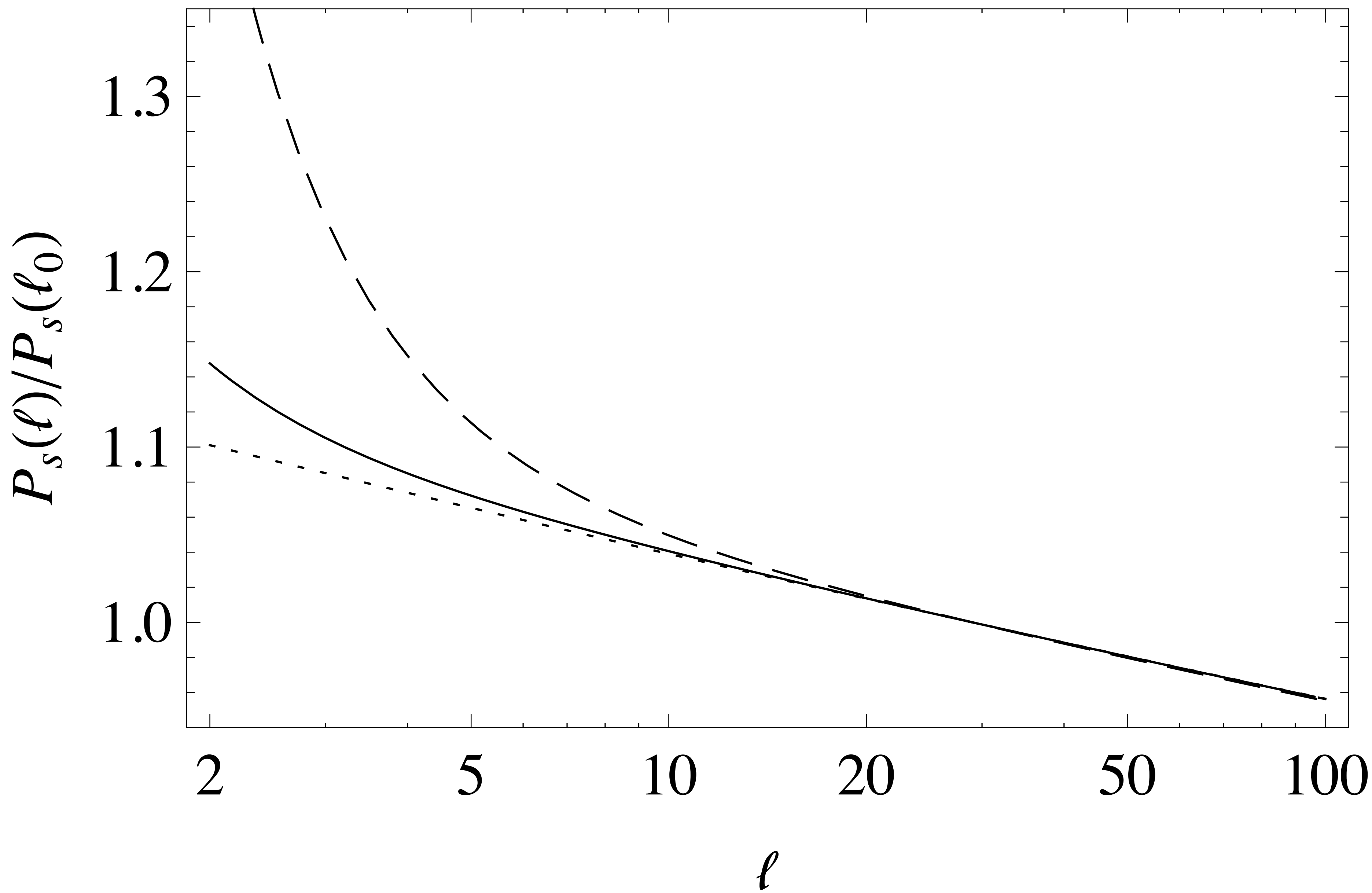
➡ comparison with other approaches to Quantum Gravity (LQG: Bojowald et al. ’11)

References

- C. Kiefer and M. K., *Quantum Gravitational Contributions to the Cosmic Microwave Background Anisotropy Spectrum*, Phys. Rev. Lett. **108**, 021301 (2012).
- C. Kiefer and M. K., *Can effects of quantum gravity be observed in the cosmic microwave background?*, Int. J. Mod. Phys. D **21**, 1241001 (2012), arXiv:1205.5161.
- C. Kiefer and T. P. Singh, *Quantum gravitational corrections to the functional Schrödinger equation*, Phys. Rev. D **44**, 1067–1076 (1991).
- J. J. Halliwell and S. W. Hawking, *The Origin of Structure in the Universe*, Phys. Rev. D **31**, 1777–1791 (1985).

Appendix: Comparison with Loop Quantum Cosmology

- LQC predicts an **enhancement** on large scales
- derivation based on inverse volume corrections



- *Figure:* Primordial power spectrum for a certain model of loop quantum cosmology (upper curve). The dotted line is the classical case, the solid line is the experimental upper bound.

➡ M. Bojowald, G. Calcagni, and S. Tsujikawa, Phys. Rev. Lett. **107**, 211302 (2011).