The first law of binary black hole mechanics

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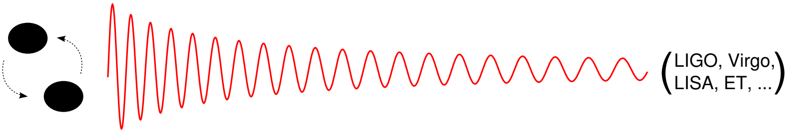
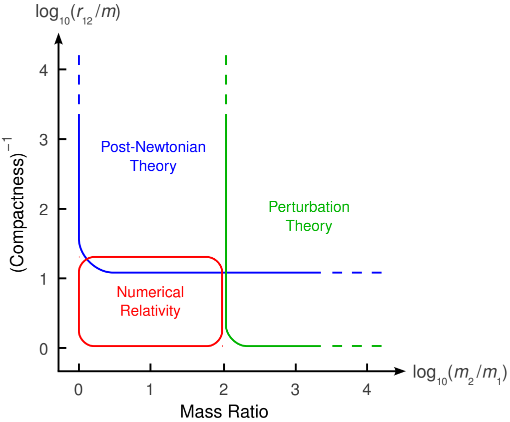


Based on collaborations with:
E. Barausse, L. Blanchet, A. Buonanno & B. F. Whiting

Methods to compute GW templates for compact binaries



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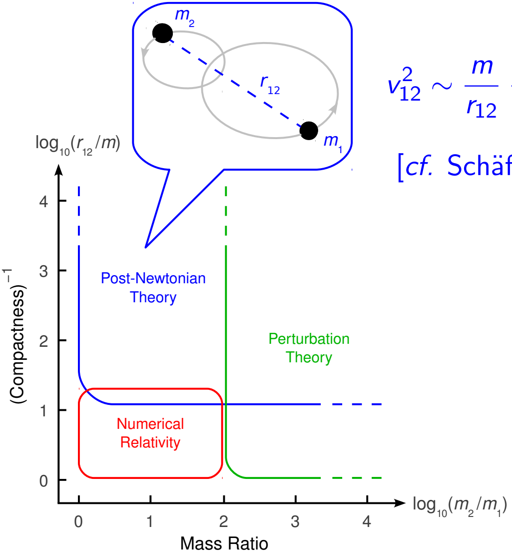
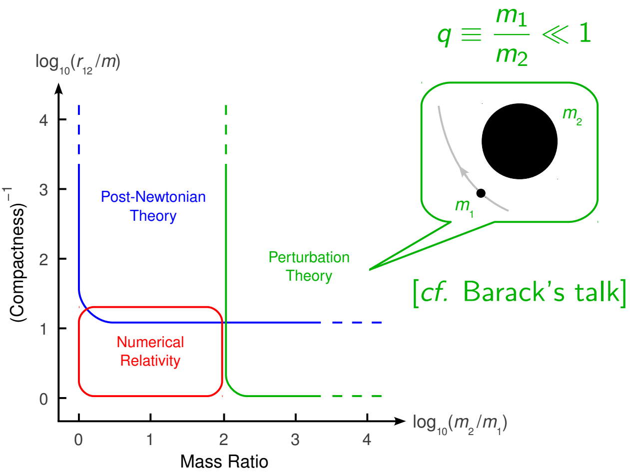


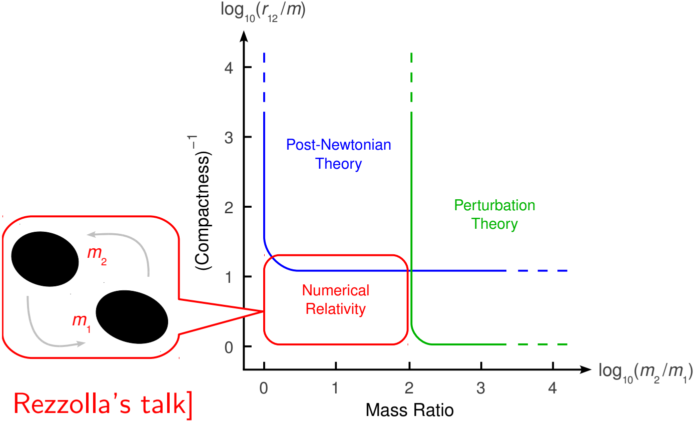
$$v_{12}^2 \sim \frac{m}{r_{12}} \ll 1$$

[*cf.* Schäfer's talk]

Methods to compute GW templates for compact binaries

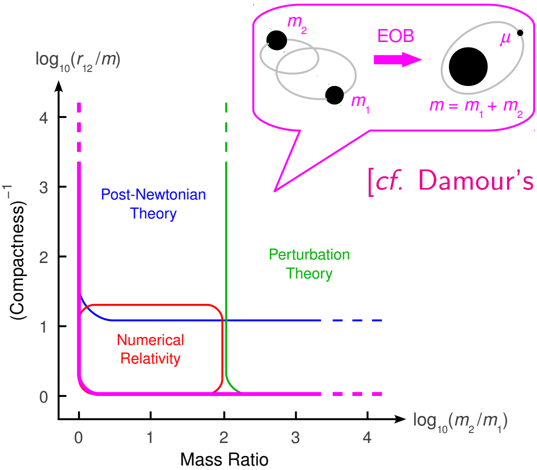


Methods to compute GW templates for compact binaries



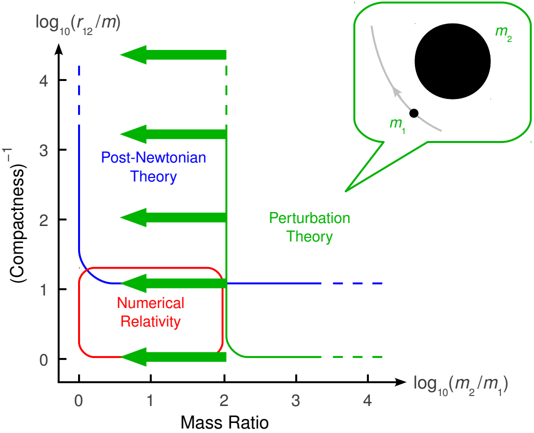
[*cf.* Rezzolla's talk]

Methods to compute GW templates for compact binaries



[*cf.* Damour's talk]

Methods to compute GW templates for compact binaries

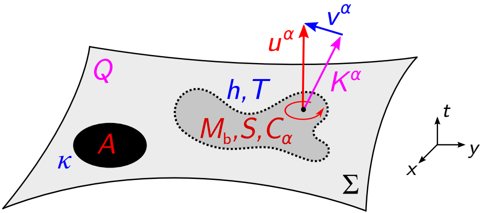


Generalized first law of mechanics

[Friedman, Uryū & Shibata, PRD (2002)]

- Spacetimes with black holes + perfect fluid matter sources
- One-parameter family of solutions $\{g_{\alpha\beta}(\lambda), u^\alpha(\lambda), \rho(\lambda), s(\lambda)\}$
- Globally defined Killing vector field K^α → conserved charge Q

$$\delta Q = \sum_i \frac{\kappa_i}{8\pi}\,\delta A_i + \int_\Sigma \left[\,\bar{h}\,\Delta(\mathrm{d}M_\mathrm{b}) + \bar{T}\Delta(\mathrm{d}S) + v^\alpha \Delta(\mathrm{d}C_\alpha)\right]$$



Application to compact binaries on circular orbits

- For circular orbits, the geometry has a helical Killing vector

$$K^\alpha \to (\partial_t)^\alpha + \Omega\,(\partial_\varphi)^\alpha \quad \text{(when } r \to +\infty\text{)}$$

- For asymptotically flat spacetimes [Friedman *et al.* (2002)]

$$\delta Q = \delta M - \Omega\,\delta J$$

- In the exact theory, helically symmetric spacetimes are not asymptotically flat [Gibbons & Stewart (1983); Klein (2004)]
- Asymptotic flatness can be recovered if gravitational radiation can be "turned off", *e.g.*
  - Conformal Flatness Condition
  - Post-Newtonian theory

Application to compact binaries on circular orbits

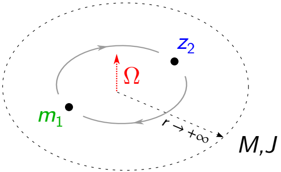
[Le Tiec, Blanchet & Whiting, PRD (2012)]

- Conservative dynamics only → no gravitational radiation
- Non-spinning compact objects modeled as point masses m_A:

$$T^{\alpha\beta} = \sum_{A=1}^{2} m_A \, z_A \, u_A^\alpha u_A^\beta \, \frac{\delta(\mathbf{x} - \mathbf{y}_A)}{\sqrt{-g}}$$

- For two point masses on a circular orbit, the first law becomes

$$\boxed{\delta M - \Omega \, \delta J = z_1 \, \delta m_1 + z_2 \, \delta m_2}$$



First integral associated with the variational law

[Le Tiec, Blanchet & Whiting, PRD (2012)]

- Variational first law: $\delta M - \Omega\,\delta J = z_1\,\delta m_1 + z_2\,\delta m_2$
- Since $\{M, J, z_A\}$ are all functions of $\{\Omega, m_A\}$, we have

$$\frac{\partial M}{\partial \Omega} = \Omega \frac{\partial J}{\partial \Omega} \quad \text{and} \quad z_A = \frac{\partial (M - \Omega J)}{\partial m_A}$$

- After a few algebraic manipulations, we obtain

$$\boxed{M - 2\Omega J = m_1 z_1 + m_2 z_2}$$

- Alternative derivations based on:
  - Euler's theorem applied to the function $M(J^{1/2}, m_1, m_2)$
  - The combination $M_\mathrm{K} - 2\Omega J_\mathrm{K}$ of the Komar quantities

Verification of the first law in PN theory

[Le Tiec, Blanchet & Whiting, PRD (2012)]

- The PN results for $M(\Omega, m_A)$, $J(\Omega, m_A)$ and $z_A(\Omega, m_A)$ are expressed in terms of

$$m \equiv m_1 + m_2\,, \quad \nu \equiv m_1 m_2/m^2 \equiv \mu/m\,, \quad \text{and} \quad x \equiv (m\Omega)^{2/3}$$

- For instance, the binding energy $E \equiv M - m$ reads

$$E = -\frac{1}{2}\,\mu\,x\left\{1 + \left(-\frac{3}{4} - \frac{\nu}{12}\right)x + \cdots + \frac{448}{15}\,\nu\,x^4 \ln x + \cdots\right\}$$

- The first law is satisfied up to 3PN order included, as well as by the 4PN+5PN logarithmic terms:

$$\frac{\partial M}{\partial \Omega} = \Omega \frac{\partial J}{\partial \Omega} \quad \text{and} \quad z_A = \frac{\partial (M - \Omega J)}{\partial m_A}$$

Binding energy beyond the test-mass approximation

[Le Tiec, Barausse & Buonanno, PRL (2012)]

- In the "small" mass ratio limit $\nu \to 0$:

$$z_1 = \sqrt{1-3x} + \nu\, z_{\text{GSF}}(x) + \mathcal{O}(\nu^2)$$

$$\frac{E}{\mu} = \left(\frac{1-2x}{\sqrt{1-3x}} - 1\right) + \nu\, E_{\text{GSF}}(x) + \mathcal{O}(\nu^2)$$

- The self-force contribution $z_{\text{GSF}}(x)$ is known numerically
  [Detweiler (2008); Sago, Barack & Detweiler (2008); Shah *et al.* (2011)]

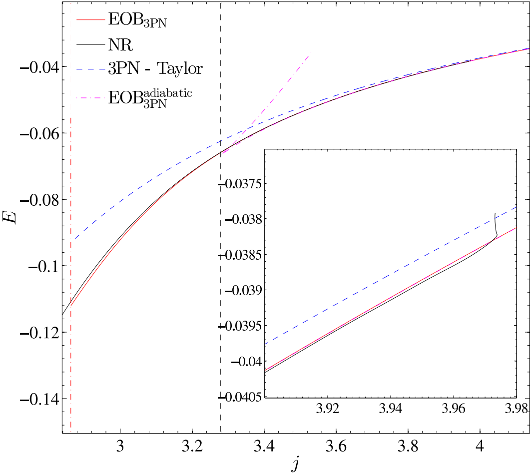
- The first law provides a relationship $E \leftrightarrow z_1$, which implies

$$E_{\text{GSF}}(x) = \frac{1}{2}\, z_{\text{GSF}}(x) - \frac{x}{3}\, z'_{\text{GSF}}(x) + f(x)$$

- A similar result holds for the angular momentum $J/(m\mu)$

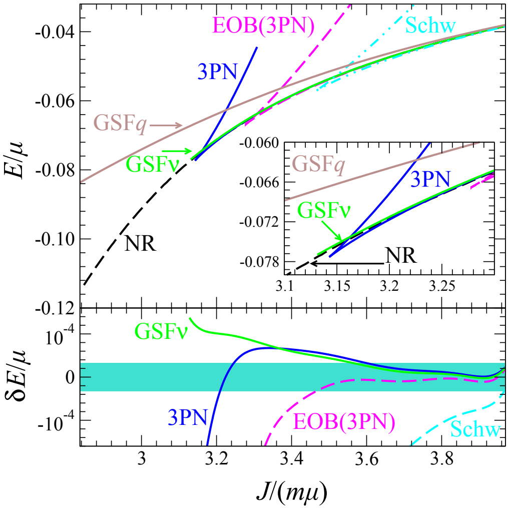
NR/EOB comparison for an equal mass binary

[Damour, Nagar, Pollney & Reisswig, PRL (2012)]



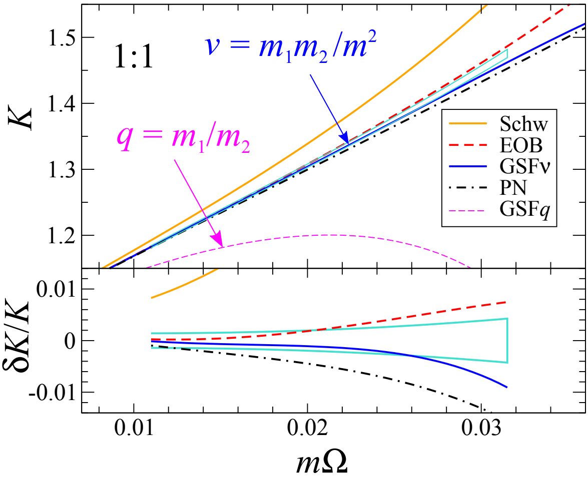
NR/GSF comparison for an equal mass binary

[Le Tiec, Barausse & Buonanno, PRL (2012)]



Periastron advance in black hole binaries

[Le Tiec, Mroué *et al.*, PRL (2011)]



Why do the GSFν results perform so well?

- In perturbation theory, one traditionally expands as

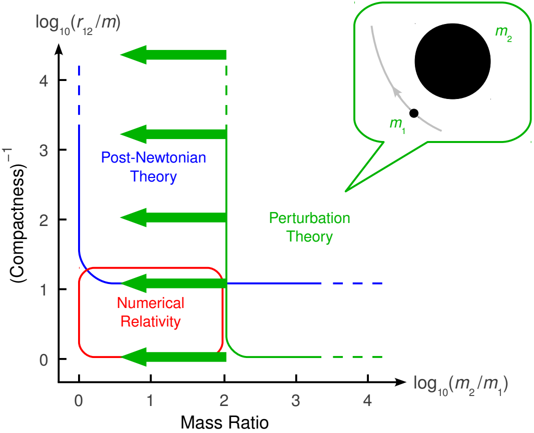
$$\text{GSF}q: \quad \sum_{n=0}^{n_{\max}} A_n(m_2\Omega)\, q^n \quad \text{where} \quad q \equiv m_1/m_2 \in [0,1]$$

- However, the relations $K(\Omega; m_A)$, $E(\Omega; m_A)$, and $J(\Omega; m_A)$ must be symmetric under exchange $m_1 \longleftrightarrow m_2$
- Hence, a better-motivated expansion is

$$\text{GSF}\nu: \quad \sum_{n=0}^{n_{\max}} B_n(m\Omega)\, \nu^n \quad \text{where} \quad \nu \equiv m_1 m_2/m^2 \in [0,1/4]$$

- In a PN expansion, we have $B_n = \mathcal{O}\big(1/c^{2n}\big) = n\text{PN} + \cdots$
- Previously noticed for head-on collision [Detweiler & Smarr (1979)]

Perturbation theory for comparable-mass binaries

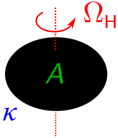


Summary and prospects

- The first law uncovers deep relations between local and global physical quantites in binary black hole spacetimes
- It holds up to very high orders in post-Newtonian theory
- Numerous applications to GW source modeling:
  - E and J at leading order beyond the test-mass results
  - Exact frequency shift of the Schwarzschild ISCO
  - EOB potentials at linear order in ν [*cf.* Barausse's talk]
  - New high-order PN coefficients in E and J
- Some directions for future research include:
  - Extending the first law to spinning point particles
  - Exploring further the use of perturbation theory to model comparable-mass compact binaries

EXTRA SLIDES

Analogies with single and binary black holes

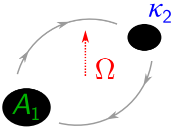


$$\delta M - \Omega_{\rm H}\,\delta J = \frac{\kappa\,\delta A}{8\pi}$$

[Bardeen *et al.* (1973)]

$$M - 2\Omega_{\rm H} J = \frac{\kappa A}{4\pi}$$

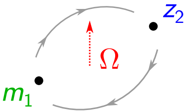
[Smarr (1973)]



$$\delta M - \Omega\,\delta J = \sum_{i=1}^{2} \frac{\kappa_i\,\delta A_i}{8\pi}$$

$$M - 2\Omega J = \sum_{i=1}^{2} \frac{\kappa_i A_i}{4\pi}$$

[Friedman *et al.* (2002)]



$$\delta M - \Omega\,\delta J = \sum_{i=1}^{2} z_i\,\delta m_i$$

$$M - 2\Omega J = \sum_{i=1}^{2} z_i m_i$$

Head-on collision of two non-spinning black holes

[Smarr (1979); Detweiler (1979)]

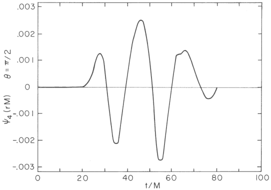


Figure 3. The curvature $\psi_4 \cdot rM$ in the equatorial plane crossing the 2-sphere at r = 25M as a function of time. This is for the two black hole collision Run II.

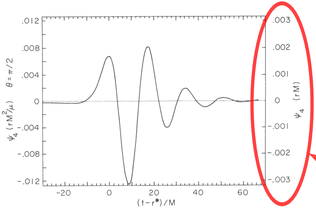
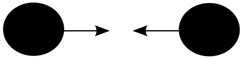


Figure 4. The same quantity as in Figure 3 except from the perturbation calculation of a particle of mass μ falling into a black hole of mass M. The abscissa is retarded time. The vertical scales are explained in the text. Only the quadrupole contribution is shown here.

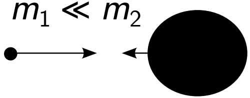
Numerical Relativity

$m_1 = m_2$



Perturbation Theory

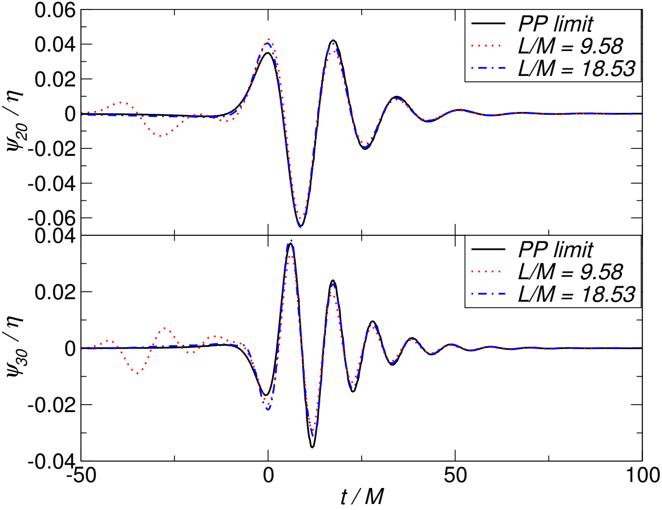
$m_1 \ll m_2$



Rescaling $m_1 \to \mu$, $m_2 \to m$

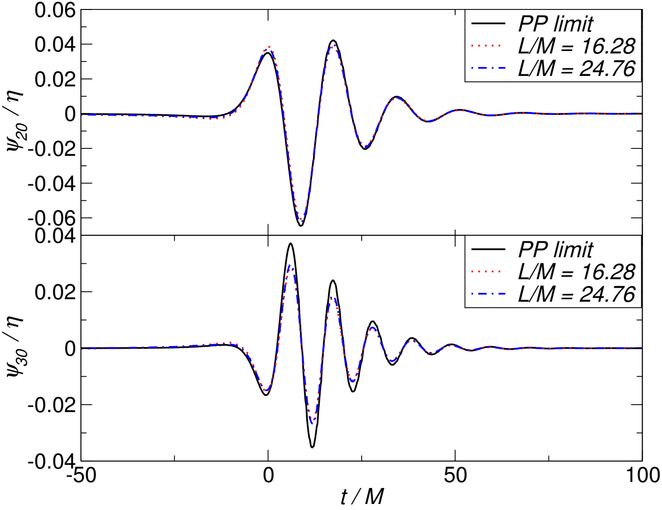
Head-on collision for a mass ratio 1:100

[Sperhake, Cardoso *et al.*, PRD (2011)]



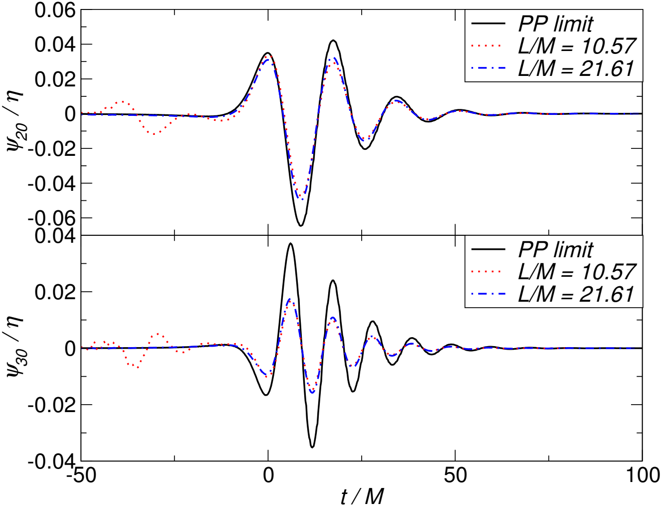
Head-on collision for a mass ratio 1:10

[Sperhake, Cardoso *et al.*, PRD (2011)]



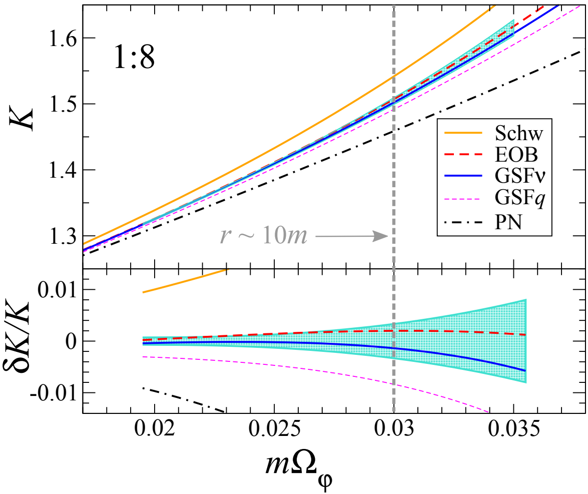
Head-on collision for a mass ratio 1:4

[Sperhake, Cardoso *et al.*, PRD (2011)]



Periastron advance for a mass ratio 1:8

[Le Tiec, Mroué *et al.*, PRL (2011)]



Variation with respect to the mass ratio

[Le Tiec, Mroué *et al.*, PRL (2011)]