Non-linear effects in non-Kerr spacetimes

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Motivation

In the case of extreme mass ratio inspirals (EMRI) systems there are mainly two reasons for studying "bumpy" black hole (non-integrable non-Kerr) spacetime backgrounds:

1) To take into account the perturbation of the supermassive black hole (SMBH) spacetime due to the accreting matter.

2) To check whether the super massive objects at the centre of galaxies are indeed Kerr black holes.

Reviews

Babak, Gair ,Petiteau & Sesana, CQG (2010)
Bambi, MPL B (2011)
Amaro-Seoane et al. ArXiv: 1202.0839

Selected papers

Ryan, PRD (1995), (1997)
Collins & Hughes, PRD (2004)
Glampedakis & Babak, CQG (2006)
Barausse, Rezzolla, Petroff & Ansorg, PRD (2007)
Gair, Li & Mandel, PRD (2008)

Metric from the Manko-Novikov family (CQG, 1992)

Weyl-Papapetrou metric element in prolate spheroidal coordinates:

$$ds^2 = g_{tt}\, dt^2 + g_{xx}\, dx^2 + g_{yy}\, dy^2 + g_{\phi\phi}\, d\phi^2 + g_{t\phi}\, dt\, d\phi$$

Gair, Li Mandel, PRD (2008)

$$\begin{aligned}
g_{tt} &= -f\,, \\
g_{xx} &= \frac{k^2 e^{2\gamma}(x^2-y^2)}{f(x^2-1)}\,, \\
g_{yy} &= \frac{k^2 e^{2\gamma}(x^2-y^2)}{f(1-y^2)}\,, \\
g_{\phi\phi} &= \left(\frac{k^2(x^2-1)(1-y^2)}{f} - f\omega^2\right)\,, \\
g_{\phi t} &= 2\,\omega\, f
\end{aligned}$$

$$\begin{aligned}
A &= (x^2-1)(1+a\,b)^2 - (1-y^2)(b-a)^2, \\
B &= [x+1+(x-1)a\,b]^2 + [(1+y)a+(1-y)b]^2, \\
C &= (x^2-1)(1+a\,b)[b-a-y(a+b)] \\
&\quad + (1-y^2)(b-a)[1+a\,b+x(1-a\,b)],
\end{aligned}$$

$$\begin{aligned}
\psi &= \beta\frac{P_2}{R^3}, \\
\gamma' &= \ln\sqrt{\frac{x^2-1}{x^2-y^2}} + \frac{3\beta^2}{2R^6}(P_3^2 - P_2^2) \\
&\quad + \beta\left(\sum_{\ell=0}^{2}\frac{x-y+(-1)^{2-\ell}(x+y)}{R^{\ell+1}}P_\ell - 2\right)
\end{aligned}$$

$$\begin{aligned}
f &= e^{2\psi}\frac{A}{B}, \\
\omega &= 2ke^{-2\psi}\frac{C}{A} - 4k\frac{\alpha}{1-\alpha^2}, \\
e^{2\gamma} &= e^{2\gamma'}\frac{A}{(x^2-1)(1-\alpha^2)^2},
\end{aligned}$$

$$\alpha = \frac{-M+\sqrt{M^2-(S/M)^2}}{(S/M)}, \qquad k = M\frac{1-\alpha^2}{1+\alpha^2},$$

$$\beta = q\left(\frac{1+\alpha^2}{1-\alpha^2}\right)^3.$$

$$\begin{aligned}
a &= -\alpha\exp\left[-2\beta\left(-1+\sum_{\ell=0}^{2}\frac{(x-y)P_\ell}{R^{\ell+1}}\right)\right], \\
b &= \alpha\exp\left[2\beta\left(1+\sum_{\ell=0}^{2}\frac{(-1)^{3-\ell}(x+y)P_\ell}{R^{\ell+1}}\right)\right], \\
R &= \sqrt{x^2+y^2-1}, \\
P_\ell &= P_\ell\left(\frac{x\,y}{R}\right)\,, \quad P_\ell(w) = \frac{1}{2^\ell \ell!}\left(\frac{d}{dw}\right)^\ell (w^2-1)^\ell
\end{aligned}$$

For q>0 the compact object is more oblate than Kerr and for q<0 more prolate.

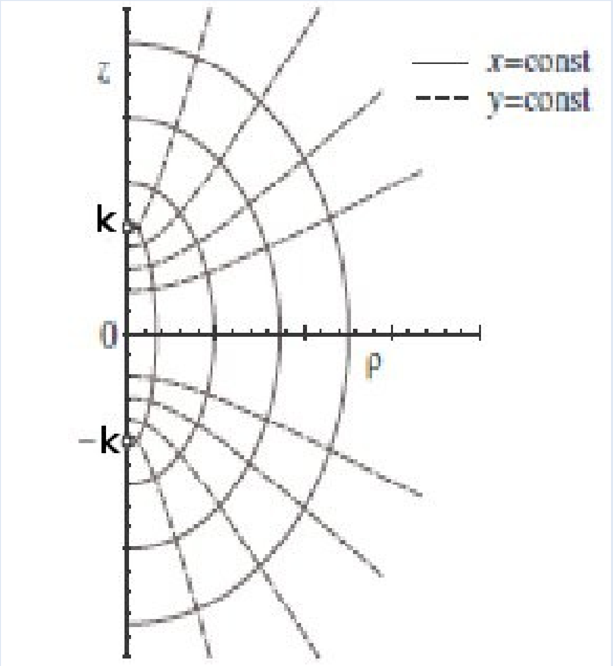
For q=0 we get the Kerr metric.

Coordinates transformations

From the prolate spheroidal coordinates (x,y) of the Weyl-Papapetrou metric to the Boyer-Lindquist (r,θ):

$r = x\,(M^2-(S/M)^2)^{1/2}+M$, $\cos\theta = y$, where $x \in [1,\infty)$, $y \in [-1,1]$

By the above transformation the event horizon lies on x=1.



From the prolate spheroidal coordinates (x,y) of the Weyl-Papapetrou metric to the cylindrical (ρ,z):

$$\rho = k\sqrt{(x^2-1)(1-y^2)}, \qquad z = kxy$$

Geodesic approximation

In the generic case of stationary and axisymmetric spacetime background, the energy E and the azimuthal angular momentum L_z are constants of motion:

$$L = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$$

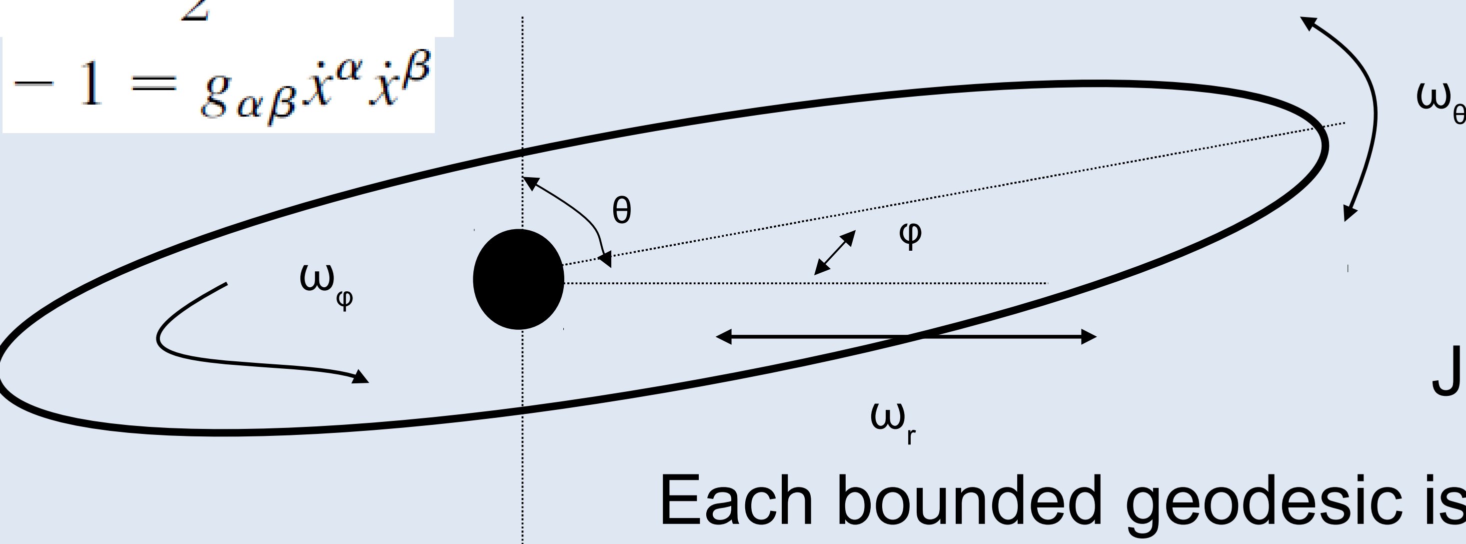
$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = g_{\mu\nu} \dot{x}^\nu$$

$$H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu$$

$$-1 = g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta$$

$$E = -g_{tt}\dot{t} - g_{t\phi}\dot{\phi}, \qquad L_z = g_{t\phi}\dot{t} + g_{\phi\phi}\dot{\phi}$$

If we had a 4th integral of motion Q, then the system would be integrable and we could express the Hamiltonian into action-angle variables $H^{(aa)}$.



$$\omega_k = \frac{\partial H^{(aa)}}{\partial J_k}$$

$J_k = J_k(H, E, L_z, Q)$

Each bounded geodesic is labelled by the 3 fundamental frequencies ω_r, ω_θ, ω_φ.

Kerr metric

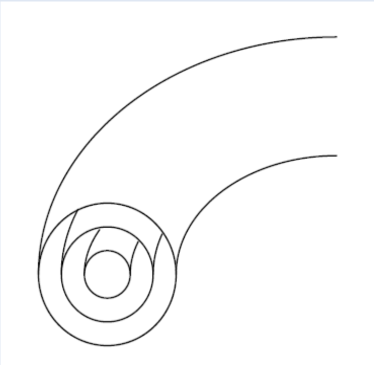
The Kerr metric has the 4th integral of motion Q, the Carter constant. Carter, PR (1968)

Though the Hamiltonian has not been yet expressed in action-angle variables, the characteristic frequencies ω_r, ω_θ, ω_φ were found "analytically" as functions of:

the black hole spin a and mass M,

the constants of motion (E, L_z, Q) and

the geometrical characteristics of the orbit (turning points of r and θ). Schmidt, CQG (2002)



The ratio $\nu_\theta(E,L_z)= \omega_r/\omega_\theta$ seems to be strictly monotonic function along a foliation of invariant tori. (numerical indications, not proven!)

Shaken, not stirred

Consider a weak perturbation of a Kerr background that "breaks" the Carter constant, but leaves the background stationary and axisymmetric, e.g. a compact object more prolate or oblate than Kerr black hole.

$$H_{New}=H_{Kerr}+q\ H_{Perturbation} \qquad (q<<)$$

By the two remaining integrals of motion E, L_z we can reduce the number of degrees of freedom from 4 to 2, i.e. instead of 4 coupled ODEs we can study 2 coupled ODEs.

ODEs: $$\frac{D\dot{x}^\lambda}{D\tau} \equiv \ddot{x}^\lambda + \Gamma^\lambda{}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0$$

Thus, we restrict the motion and the dynamics on the meridian plane ((r,θ) Boyer-Lindquist, (ρ,z) Weyl-Papapetrou).

Two theorems to rule them all

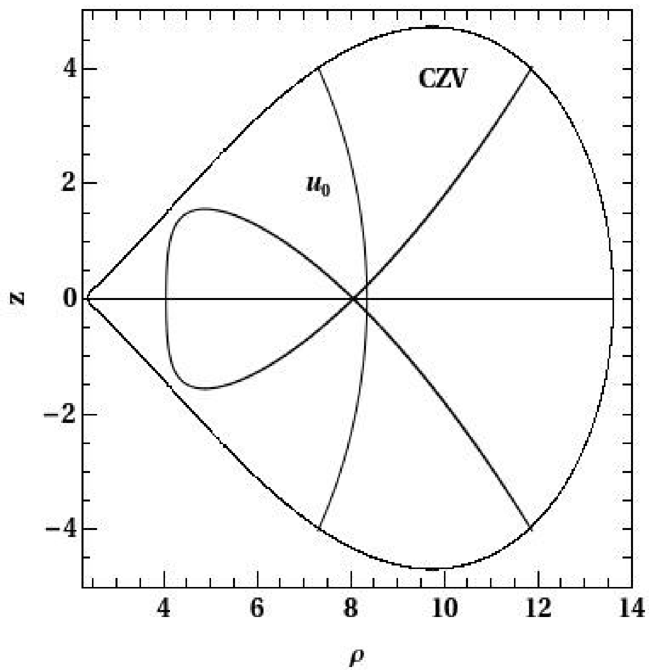
If the perturbation is weak enough, then most of the non-resonant tori survive (KAM theorem) and the resonant tori transform into the Birkhoff chain (Poincaré-Birkhoff theorem).

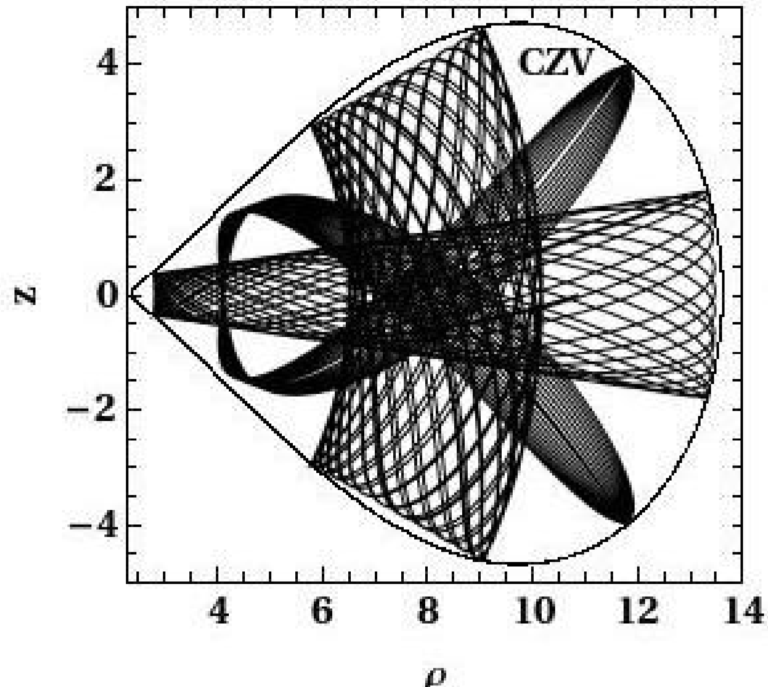
Resonance condition:

$$\sum_{i=1}^{n} k_i \omega_i = 0,\ k_i \in \mathbb{Z},\quad |k| = \sum_{i=1}^{n} |k_i| \neq 0 \quad .$$

n=2

Regular Orbits on a Poincaré section

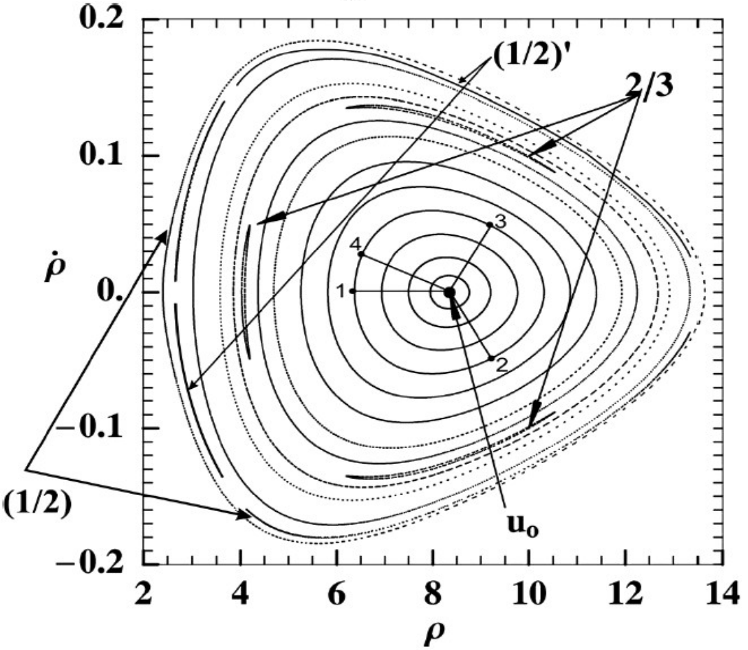


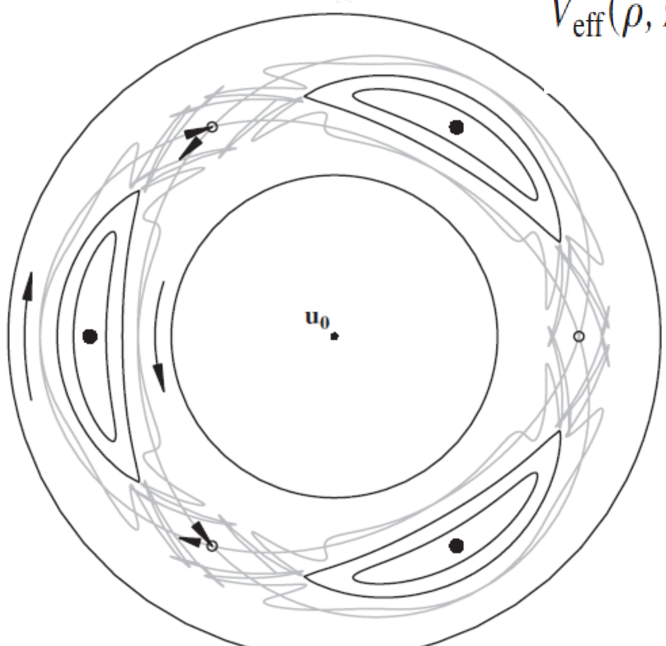


Curve of Zero Velocity (CZV)

$$\frac{1}{2}(\dot{\rho}^2 + \dot{z}^2) + V_{\text{eff}}(\rho, z) = 0,$$

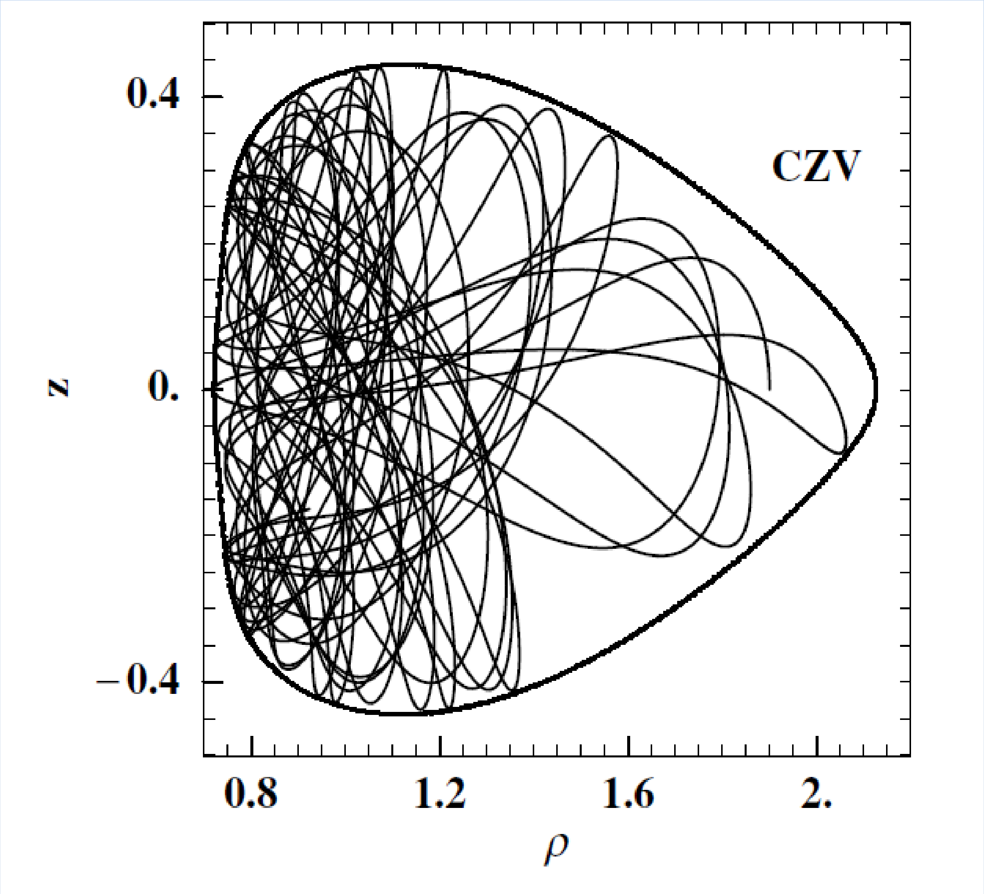
$$V_{\text{eff}}(\rho, z) = \frac{1}{2} e^{-2\gamma}\left[f - E^2 + \left(\frac{f}{\rho}(L_z - \omega E) \right)^2 \right]$$

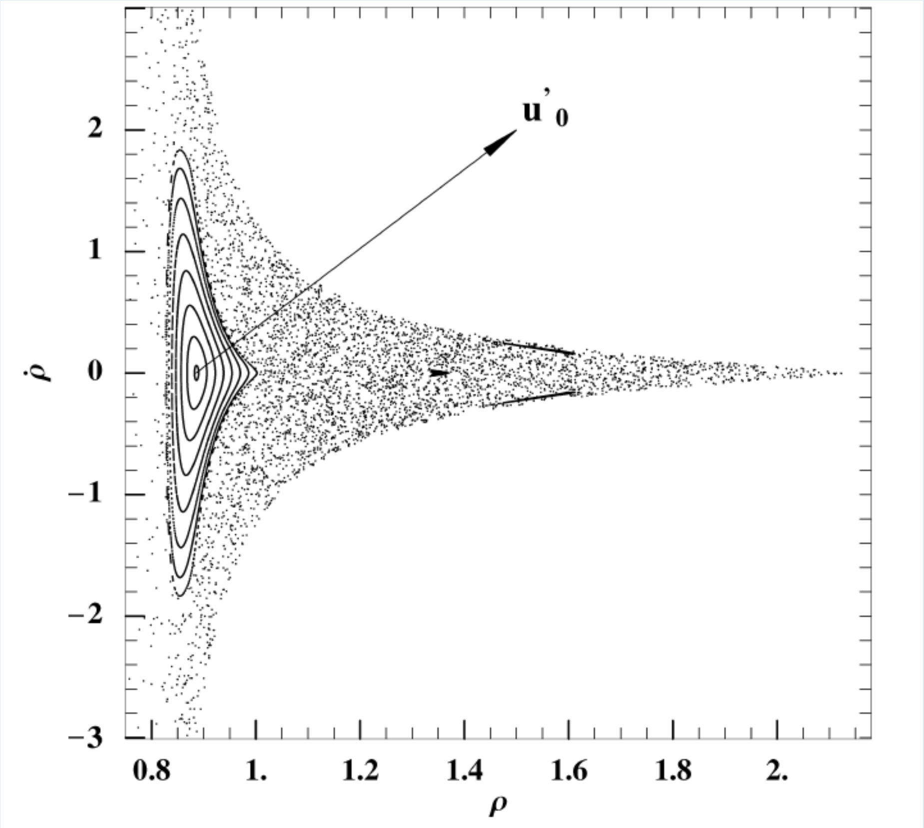




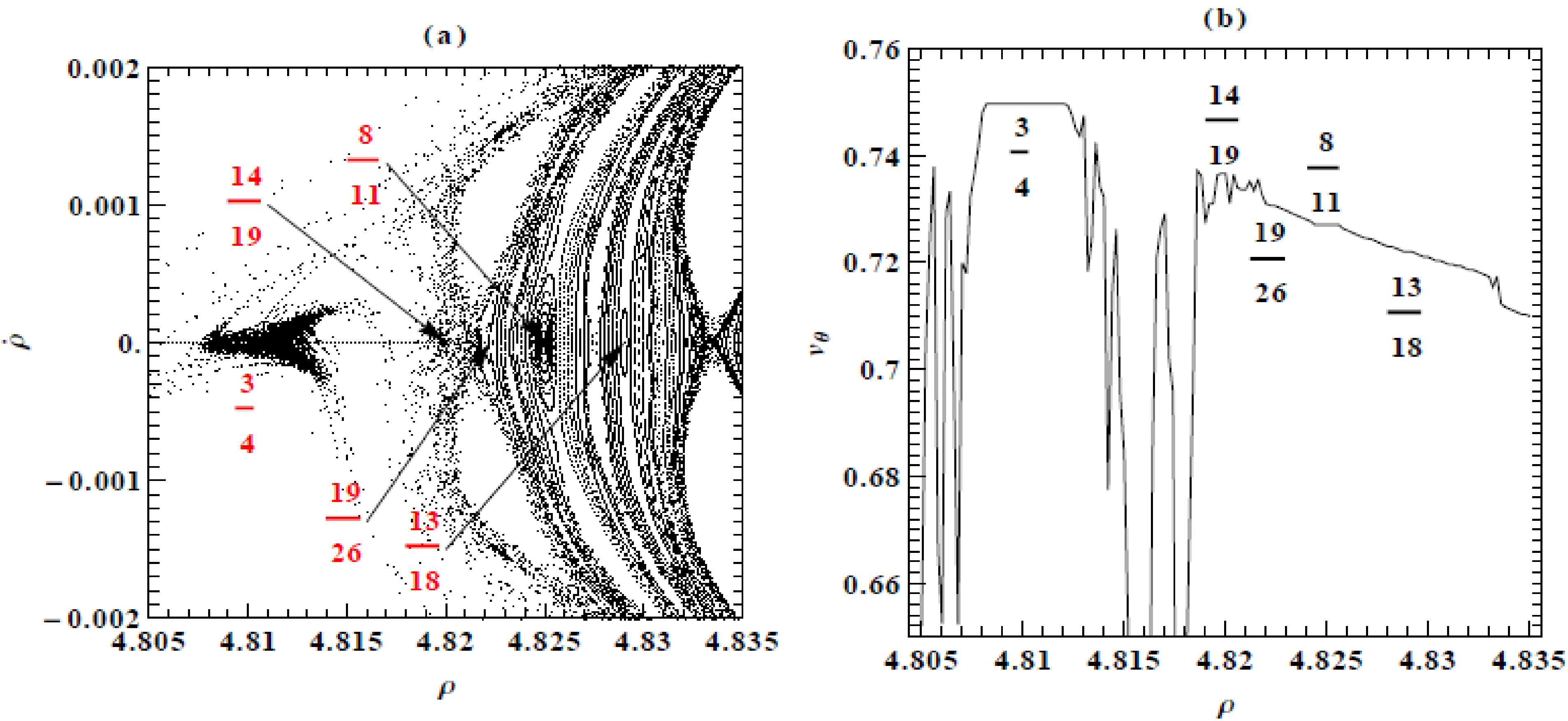
Poincaré Section
z=0, dz/dτ>0

Chaotic Orbits on a Poincaré section





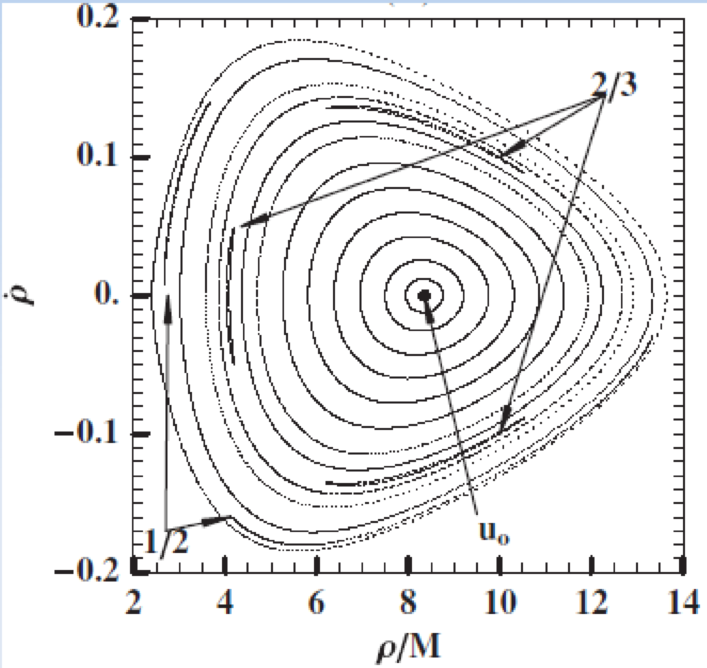
The Rotation Number

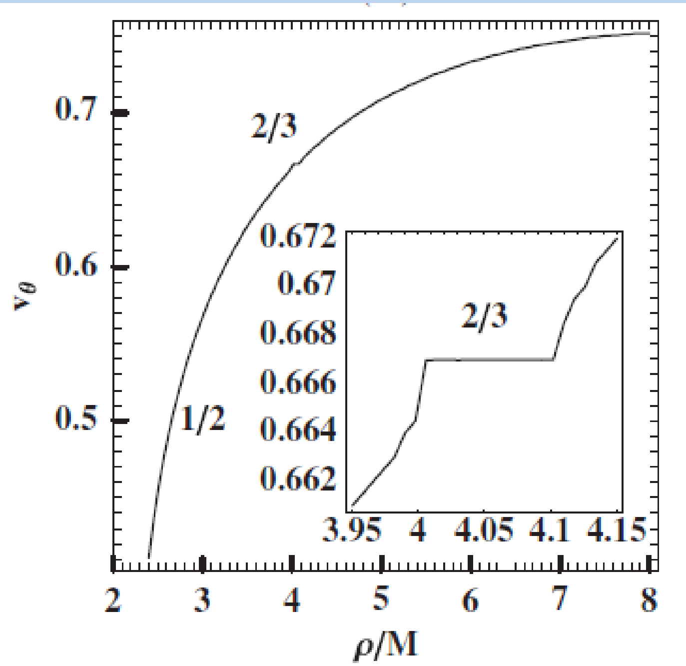


L-G, “The non-integrability of the Zipoy-Voorhees metric”, arXiv:1206.0660

The rotation number v_θ is the ratio of two fundamental frequencies of the system.

Effect of resonances



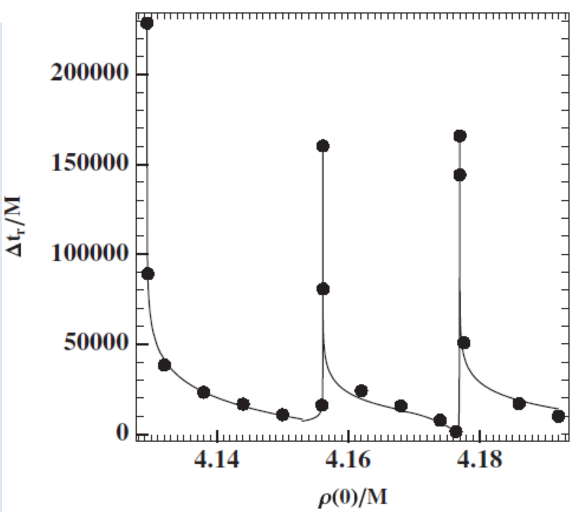


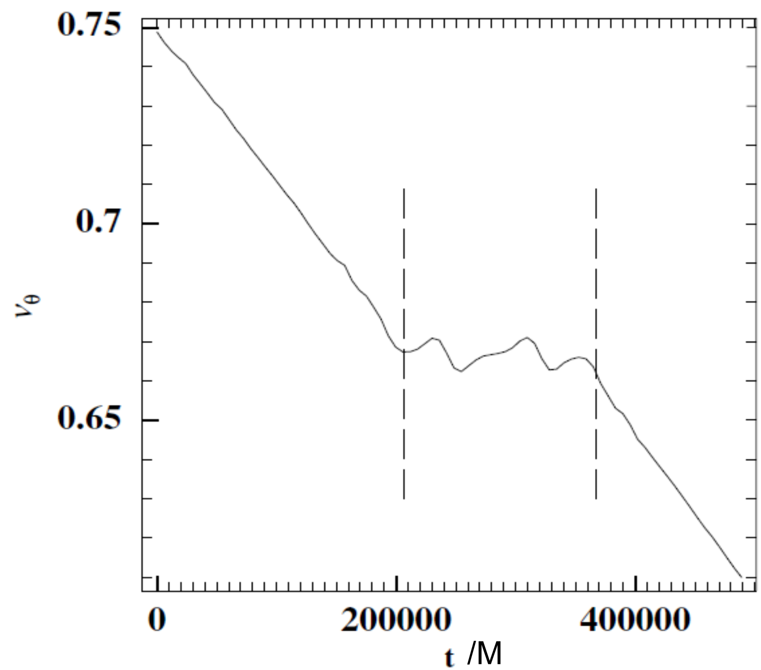
We modified the rates of energy and angular momentum loss proposed by Gair & Glampedakis (PRD, 2006) for a Kerr black hole according to the prescription given in Gair, Li & Mandel, PRD (2008)

$$E(t) = E(0) + \left.\frac{dE}{dt}\right|_0^t$$

$$L_z(t) = L_z(0) + \left.\frac{dL_z}{dt}\right|_0^t$$

The adiabatic approximation (radiation reaction included)

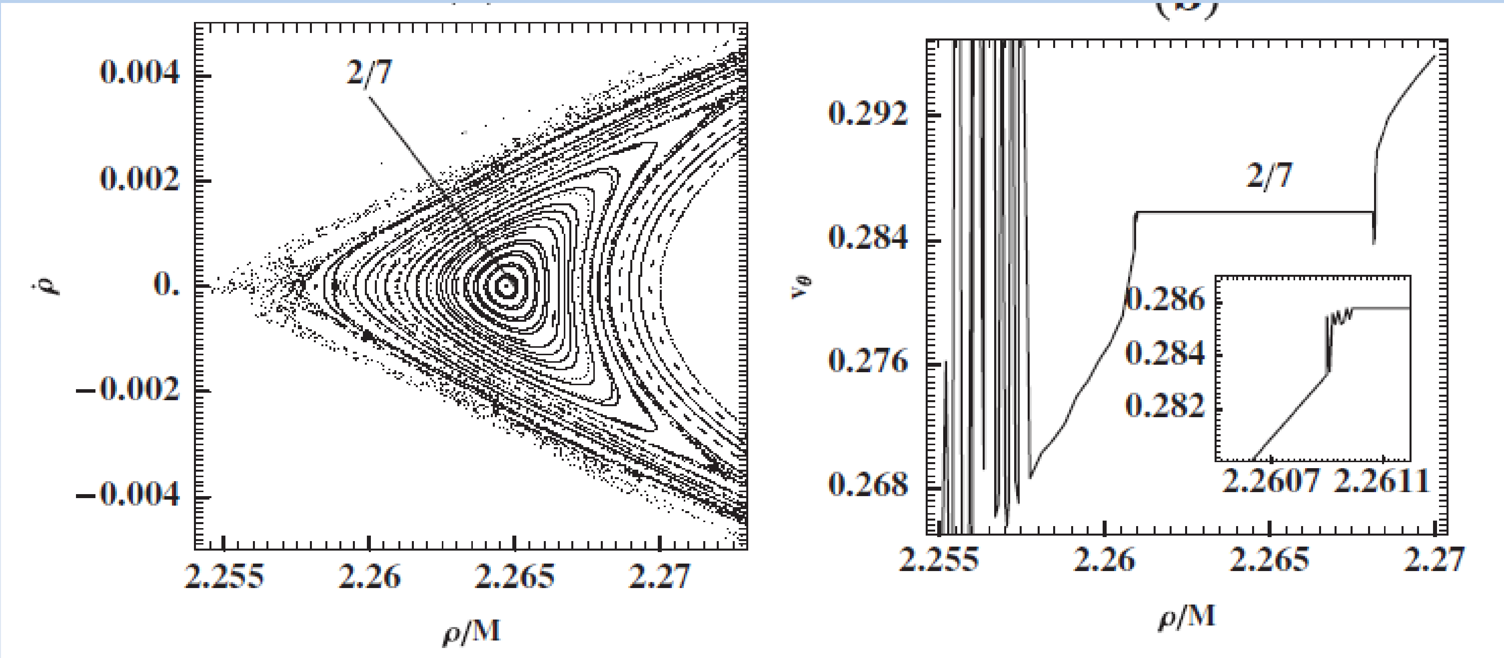




The unit of time t is 5 M/M_{Sun} μs. For $M=10^6 M_{Sun}$ a resonance that endures $\Delta t=5\ 10^4$ M corresponds approximately to a week.

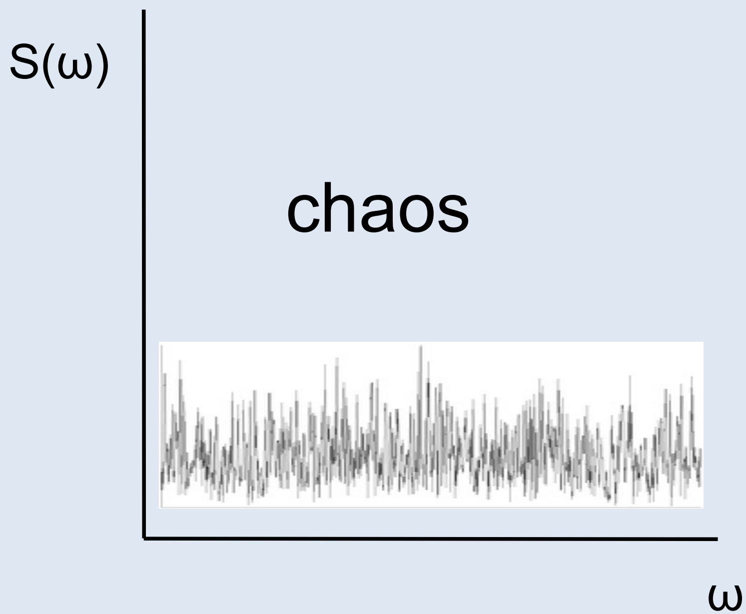
Apostolatos, L-G & Contopoulos, PRL (2009)
L-G, Apostolatos & Contopoulos, PRD (2010)
Contopoulos, L-G & Apostolatos, IJBC (2011)

Beacon effect of stickiness



The stickiness concerns chaotic orbits which for various reasons stick for a long time interval in a region, close to an invariant curve, so that their behaviour may resemble that of regular orbits, before extending further away.

Contopoulos, Order and Chaos in Dynamical Astronomy, Springer (2002)



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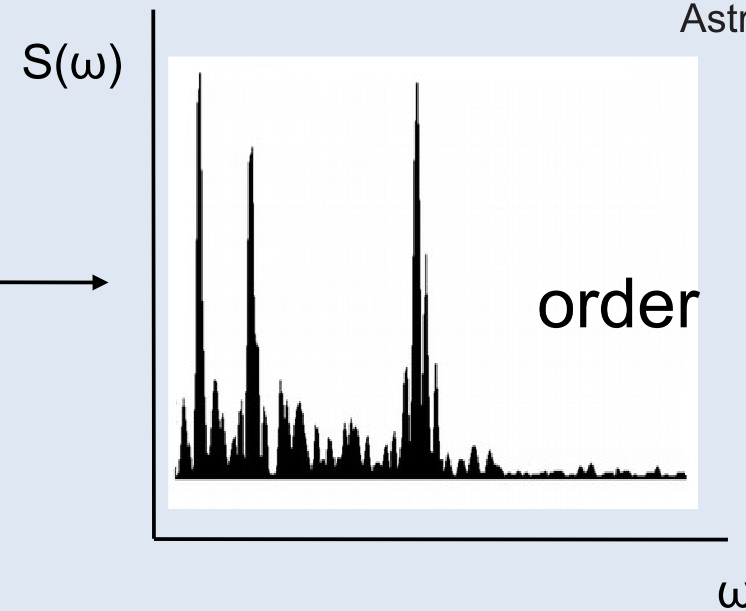
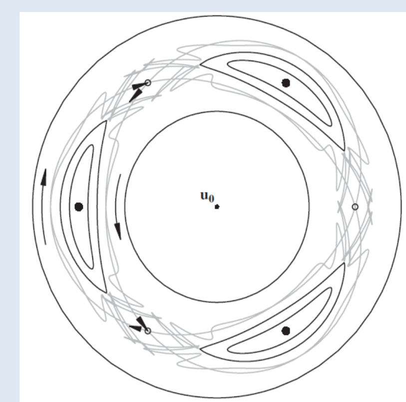
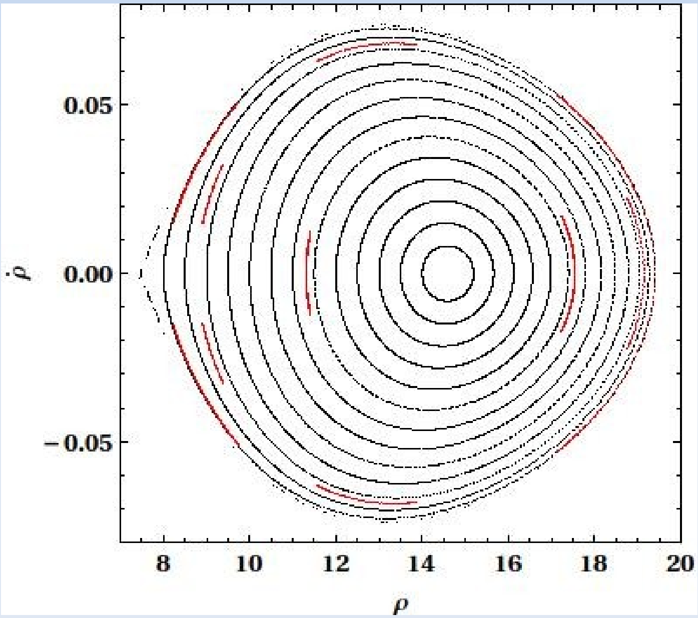


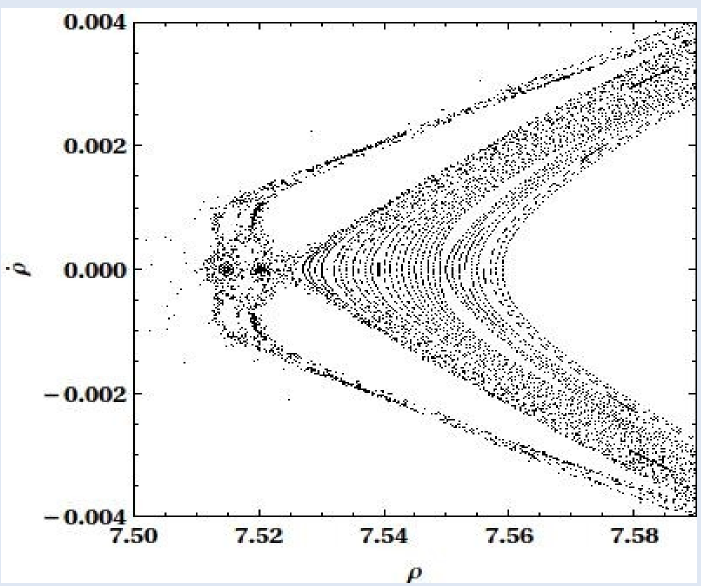


L-G, Apostolatos & Contopoulos, PRD (2010)

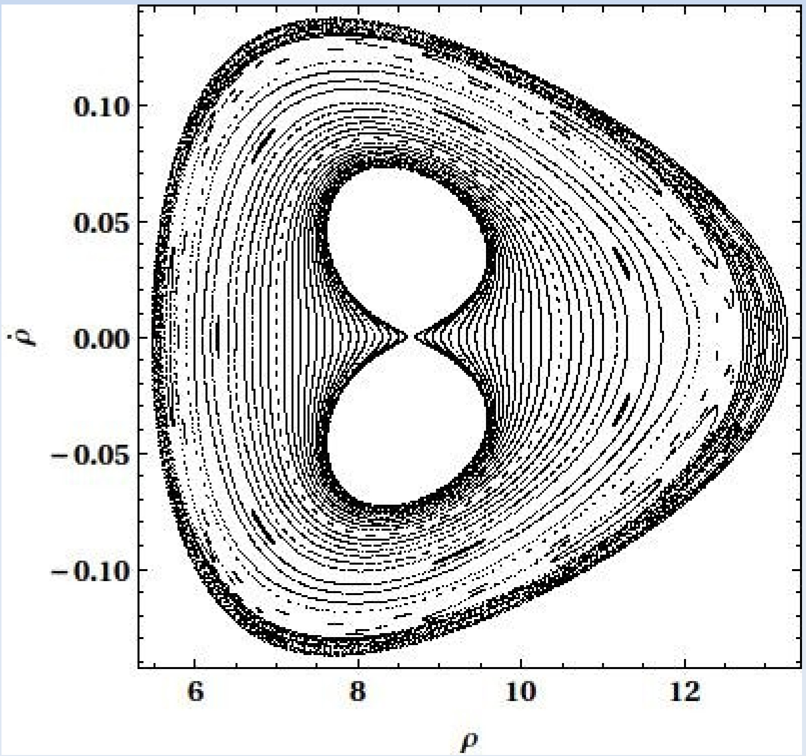
Different metrics similar structures

Zipoy, JMP (1966) Voorhees, PRD (1970) metric





Manko, Sanabria-Gómez, Manko, PRD (2000) metric



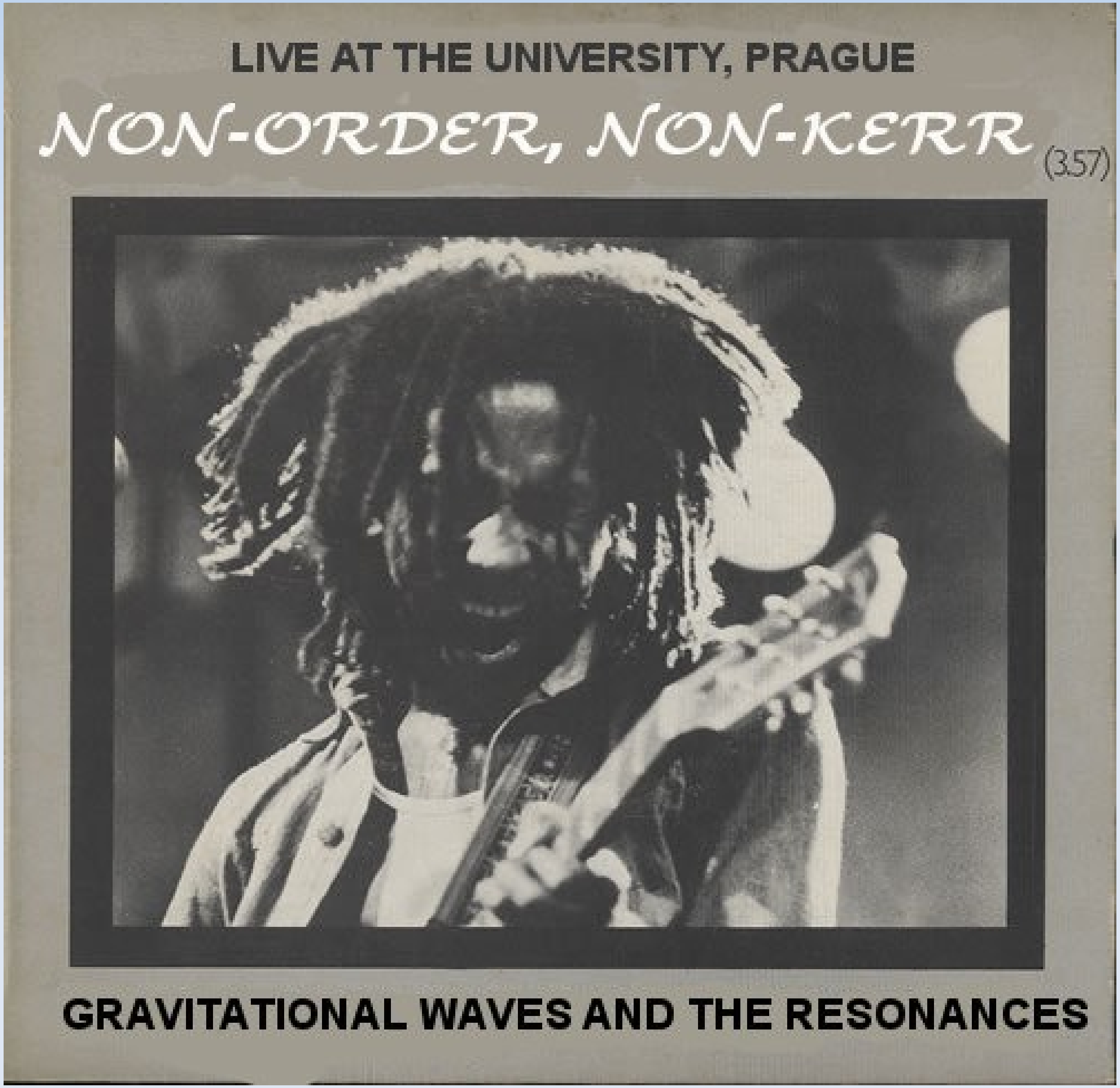
Seyrich, L-G, in preparation

L-G, arXiv:1206.0660

Conclusions

The resonance and the stickiness effect are generic characteristic of the geodesic motion in <u>any</u> non-integrable Hamiltonian describing a stationary and axisymmetric background, i.e. in <u>any</u> (weak) perturbation of Kerr spacetime which remains stationary and axisymmetric.

Thank you!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NON-ORDER, NON-KERR (3.57)
GRAVITATIONAL WAVES AND THE RESONANCES