The conformal Einstein field equations for trace-free perfect fluids

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Presenting results of joint work with Juan A. Valiente Kroon

Cosmology and stability

Central question

Any model is an approximation of particular features of our universe.
How sensitive are predictions using these models to perturbations?

Stability results using functional analysis (examples):

- **vacuum:** Christodoulou-Klainermann [1993], Anderson [2005], Lindblad-Rodnianski [2010]
- **EM:** Zipser [2000], Loizelet [2008]
- **scalar field:** Ringström [2008], Holzegel, Smulevici [2011]
- **perfect fluid:** Rodnianski-Speck [2010], Speck [2011, 12]

Stability results using the conformal methods:

- **vacuum:** Friedrich [1981, 85, 86] LV [2010, 11]
- **EMYM:** Friedrich [1991], LV [2012] (EM)

Main theorem

We are interested in the stability of perfect fluid FLRW space-times with an equation of state $\tilde{p} = \frac{1}{3}\tilde{\rho}, \ (\gamma = \frac{4}{3})$. These space-times describe *incoherent radiation* and have a trace-free energy-momentum tensor.

In particular we prove:

Theorem (LV 2012)

Suppose we are given Cauchy initial data for the Einstein-Euler system with a de Sitter-like cosmological constant λ and equation of state $\tilde{p} = \frac{1}{3}\tilde{\rho}$.
If the initial data is sufficiently close to data for a FLRW cosmological model with $\tilde{p} = \frac{1}{3}\tilde{\rho}$, cosmological constant λ and spatial curvature $k = 1$, then

- *the development exists globally towards the future,*
- *is future geodesically complete,*
- *remains close to the FLRW solution.*

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Conformal approach of Friedrich

Outline of the general idea

- Conformally embed $(\tilde{M}, \tilde{g})$ into (M, g) where $g = \theta^2 \tilde{g}$.
- Re-formulate the Einstein field equation in terms of the geometry of (M, g).
- Study global problems in $(\tilde{M}, \tilde{g})$ via local analysis in (M, g)
- Show regularity of PDE and formulate evolution problem.
- Prove existence and uniqueness.
- Give reference space-time and prove stability following Kato [1975].

For more details see the plenary talk by H.Friedrich on Wednesday.

- The CEFE for trace-free matter models were discussed by Friedrich [1991]
- The equations contain $T_{ij} = \theta^{-2} \tilde{T}_{ij}$ and its first derivative
- The details for setting up the required first order symmetric hyperbolic (FOSH) system depend on the matter model itself - EMYM discussed in Friedrich [1991]

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The conformal variables: Geometry

Following Friedrich [1991]

Geometry:

- **coordinates:** $(\tau, x^{\mathcal{A}})$
- **Frame:** g-orthonormal ($g_{\mu\nu}$ indirectly defined via frame metric) $\rightsquigarrow$ 1+3 split and space-spinors
- **Connection:** here Levi-Civita connection for g
- **Curvature:** decomposed into Weyl and Schouten tensor
- **conformal factor:** and its derivatives

Conformal factor θ must satisfy $\nabla^k \nabla_k \theta = \theta P$. Fix θ by setting $P = -1$

Additional gauge source functions chosen to *fix gauge freedoms* of coordinates and frame and give symmetric hyperbolic system (Friedrich 1985).

The conformal variables: Matter

Matter (radiation fluid $\gamma = \frac{4}{3}$) trace-free:

$$\tilde{T}_{ij} = \frac{4}{3}\tilde{\rho}\tilde{u}_i\tilde{u}_j - \frac{1}{3}\tilde{\rho}\tilde{g}_{ij}$$

Define new variables:

- **density:** $\rho = \frac{\tilde{\rho}}{\theta^4}$
- **fluid flow:** $u_i = \theta\tilde{u}_i$ with $g(u,u) = 1$

$$\Rightarrow \quad T_{ij} = \tfrac{4}{3}\rho u_i u_j - \tfrac{1}{3}\rho g_{ij} = \theta^{-2}\tilde{T}_{ij}$$

- **derivatives of the above:** $\nabla_i\rho \to \rho_i \quad \nabla_i u_j \to u_{ij}$

Evolution equations and constraints derived from $\nabla^i T_{ij} = 0$

Following Choquet-Bruhat [2008] one obtains a FOSH system for $\rho, \rho_i, u_j, u_{ij}$

Existence and uniqueness

The equations satisfied by the geometric and matter variables are known as the *conformal Einstein field equations* (CEFE) - here for a radiation fluid - and have been formulated as a FOSH system.

Regularity of the CEFE

If $\rho > 0, u_0 \neq 0$ then the CEFE form a regular FOSH system.
In particular, this system is regular at conformal infinity $\mathscr{I}$, i.e. when $\theta = 0$.

Existence and uniqueness

Given sufficiently smooth initial data $\mathbf{w_0}$ for the (radiation fluid) CEFE on $\mathcal{U} \subset \mathbb{S}^3$ there exists a unique solution $\mathbf{w}$ in a neighbourhood of $\mathcal{U}$.

Radiation fluid space-time

A solution (M, g) to the CEFE (for a radiation fluid) implies a solution $(\tilde{M}, \tilde{g})$ to the Einstein field equations for a radiation fluid, where $\tilde{M} = M|_{\{\theta>0\}}, \tilde{g} = \theta^{-2} g$.

FLRW and the conformal reference space-time

Friedmann-Lemaitre-Robertson-Walker (FLRW) metric

$$ds^2_{FLRW} = dt^2 - \frac{a(t)^2}{(1+\frac{1}{4}kr^2)^2}(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2))$$

$$k = 1 \Rightarrow \qquad ds^2_{FLRW} = a(t)^2\left[d\tau^2 - d\sigma^2_{\mathbb{S}^3}\right] = a(t)^2 ds^2_{EC}$$

where $\tau = \int_{t_0}^{t} \frac{dt'}{a(t')}$ and $d\sigma^2_{\mathbb{S}^3}$ is the standard metric on $\mathbb{S}^3$.

- Work with $(M, g) = (\mathbb{R} \times \mathbb{S}^3, g_{EC})$ and $(\tilde{M}, \tilde{g}) = (I \times \mathbb{S}^3, g_{FLRW})$.
- For $\lambda < 0$ (deSitter-like case), $\mathscr{I}^+$ is a space-like hypersurface $\tau = \tau_\infty(x^{\mathcal{A}})$.
- Use FLRW with $\gamma = \frac{4}{3}$, $k = 1$, $\lambda < 0$ as reference space-time and read off initial data $\mathring{\mathbf{w}}_0$ for the CEFE (note $P_{EC} = -1$).

Stability theorem for a tracefree perfect fluid

We work in $H^m(\mathbb{S}^3, \mathbb{R}^N)$, where $m > 4$

Theorem

Let $\mathbf{w}_0$ be initial data on $\mathbb{S}^3$ for CEFE for radiation fluids with $\lambda < 0$ such that $\mathbf{w}_0$ is sufficiently close to $\mathring{\mathbf{w}}_0$ (FLRW data with $\lambda < 0$ and $k = 1$). Then

- *a solution $\mathbf{w}$ to the CEFE exists on $[0, T] \times \mathbb{S}^3$ with $T > \tau_\infty$,*
- *$\mathbf{w}$ implies a C^{m-2} solution of the Einstein equations for a radiation fluid on $\tilde{M} = \{p \in [0, T] \times \mathbb{S}^3 : \theta(p) > 0\}$,*
- *the development exists globally towards the future,*
- *$\tilde{M}$ is future geodesically complete and $\mathscr{I}^+$ a space-like hypersurface.*
- *$\mathbf{w}$ remains close to the FLRW solution, which is hence non-linearly stable.*

Remarks:

- The results complement [Speck 2011], where $\gamma \neq \frac{4}{3}$.

Future work and open questions

1. *Can a similar result be obtained for null dust?*
2. *Can one use congruences to fix the gauge choice?* $\rightsquigarrow$ Weyl connections
   - conformal geodesics for vacuum [Friedrich 1995, 2003, LV 2010, 2011]
   - conformal curves for Einstein-Maxwell [LV 2012]
3. *Can one use congruences to locate conformal infinity?*
   Explicit knowledge of the conformal factor in terms of the time parameter of the above curves allows to prescribe / predict the location of the conformal infinity.

Thank you for listening

Reference: arXiv:1111.4691