Finsler spacetimes and gravity: a consistent extension of general relativity

Christian Pfeifer and Mattias N.R. Wohlfarth, II. Institute for theoretical physics, University of Hamburg; Contact: christian.pfeifer@desy.de

Motivation

For a hundred years Lorentzian manifolds serve as geometric background for physics.

Equipped with the standard model of particle physics this led to the explanation of a huge amount of observations. However, on this basis we have to conclude that 96% of the universe are unknown; called dark matter and dark energy [1]. Today most explanation attempts for this fact come from particle physics, but possibly a well controlled extension of the geometric background for physics can shed light on the dark universe.

Here we present Finsler spacetimes which are capable to serve as generalized geometric background for physics providing:

I. CAUSALITY in a precise defined way,
II. OBSERVERS and their measurements,
III. FIELD THEORIES and
IV. GRAVITATIONAL DYNAMICS consistent with general relativity.

This invitation is based on our articles [3,4].

Finsler geometry

One of the fundamental measurements in physics is the measurement of time.

Its theoretical description is given by Einstein's clock postulate: An observer on worldline x[τ] measures the time

$$S[x]=\int d\tau\sqrt{|g_{ab}(x)\dot{x}^a\dot{x}^b|}\,.$$

The fundamental object is the metric g, which determines the metric geometry of spacetime.

The key idea for Finsler spacetimes is a more general description of the measurement of time which also realizes the weak equivalence principle:

$$S[x]=\int d\tau F(x,\dot{x})\,.$$

It is based on a one-homogeneous function F on the tangent bundle which determines the Finsler geometry of spacetime [2].

Finsler geometry equals metric geometry in the case F is given by the metric length measure used in the Einstein clock.

I Causality

The description of Finsler spacetimes requires the tangent bundle TM of the spacetime manifold M. We consider it in manifold induced coordinates

$$(x,y)=Z\in TM,\ Z=y^a\partial_{a|x}\,.$$

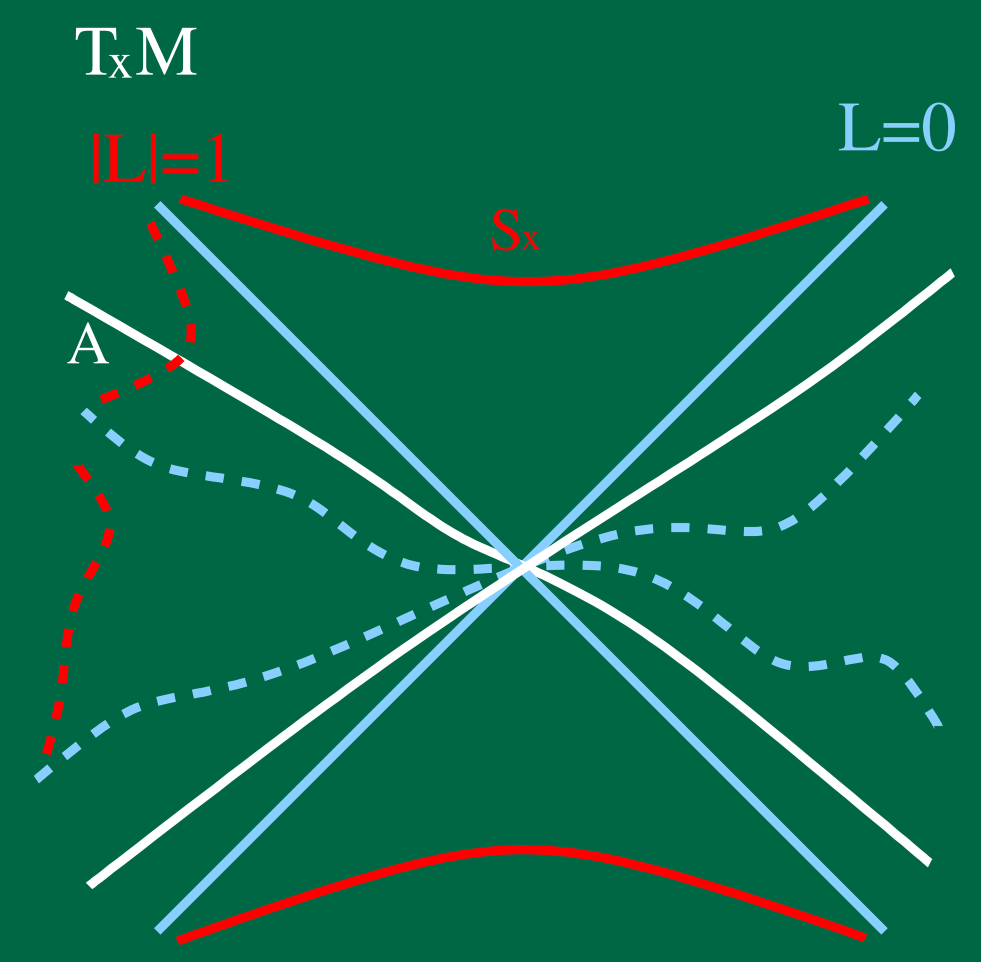
and the corresponding basis of TTM

$$\{\partial_a=\frac{\partial}{\partial x^a},\bar{\partial}_a=\frac{\partial}{\partial y^a}\}.$$

A Finsler spacetime (M,L,F) is a smooth manifold M equipped with a continuous function L: TM → R s. th.:

- **L is smooth on TM** $\setminus\{0\}$,
- **L is reversible** $|L(x,y)|=|L(x,-y)|$,
- **L is homogeneous of degree r:** $L(x,\lambda y)=\lambda^r L(x,y)\ \forall\lambda>0$,
- $g^L_{ab}(x,y)=\frac{1}{2}\bar{\partial}_a\bar{\partial}_b L$ **is non-degenerate on TM** $\setminus$ **A; A of measure zero,**
- $\forall x\in M\ \exists$ **a non-empty closed connected component** $S_x\subset T_xM$, **where** $|L|=1$, $g^L_{ab}(x,y)$ **has signature** $(\epsilon,-\epsilon,-\epsilon,-\epsilon),\epsilon=\frac{L}{|L|}$;
- $F=|L|^{1/r},\ g^F_{ab}=\frac{1}{2}\bar{\partial}_a\bar{\partial}_b F^2$.

Our definition of Finsler spacetimes guarantees a causal structure in each tangent space: S_x is the shell of unit timelike vectors which defines a cone of timelike directions with null boundary.



The geometry of Finsler spcaetime is based on the unique Cartan non-linear connection coefficients ("Christoffel symbols") on TM

$$N^a{}_b=\frac{1}{4}\bar{\partial}_b(g^{Laq}(y^m\partial_m\bar{\partial}_q L-\partial_q L)).$$

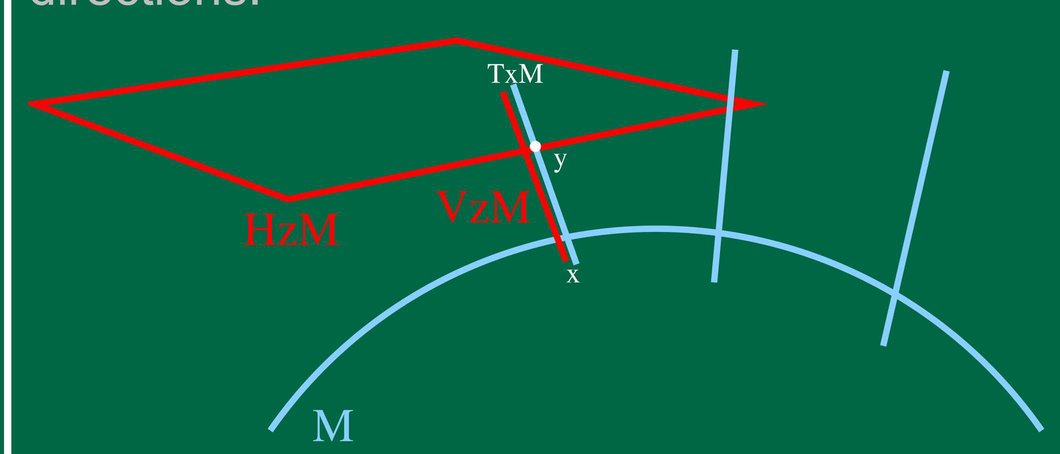
Theorem: Everywhere where L and F are both differentiable they encode the same geometry, i.e. N[L]=N[F²]!

II Observers

The nonlinear connection coefficients split TTM and T*TM into horizontal and vertical space by

$$\{\delta_a=\partial_a-N^q{}_a\bar{\partial}_q,\bar{\partial}_a\},$$
$$\{dx^a,\delta y^a=dy^a+N^a{}_q dx^q\}\,.$$

The horizontal space is identified with the (co-)tangent space along the manifold directions.



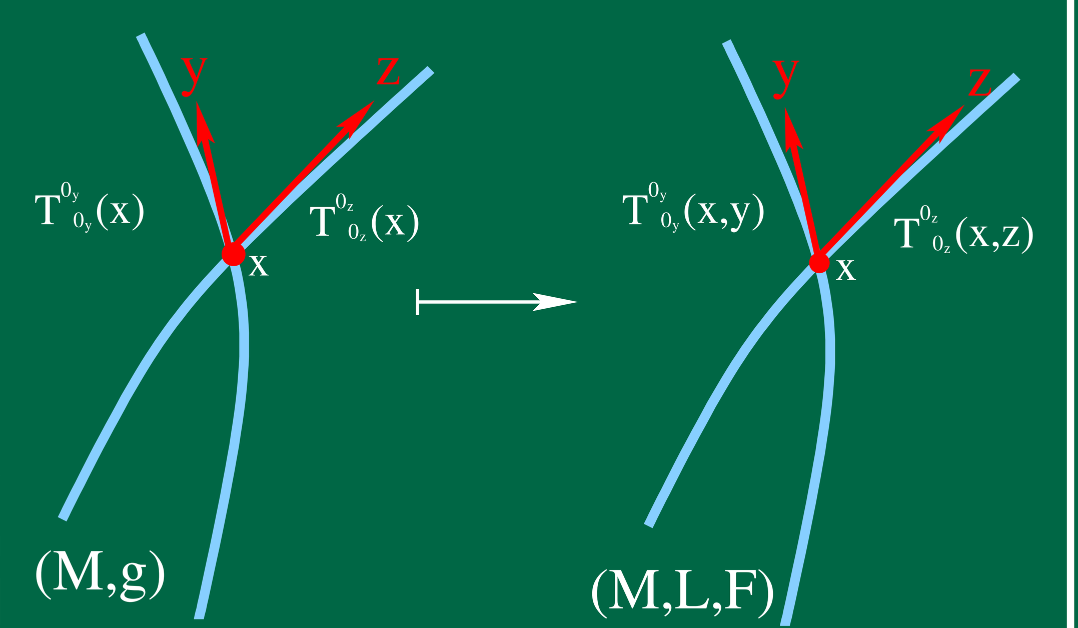
Observers are moving on worldlines x[τ] on M with trajectory (x[τ], ẋ[τ]) on TM where ẋ[τ] lies in the cone of timelike vectors.

They are equipped with an horizontal orthonormal frame defining their time and space directions.

$$\{E_a\}=\{E_0=\dot{x}^a\delta_a,E_\alpha\};\ g^F_{(x,\dot{x})}(E_a,E_b)=-\eta_{ab}.$$

Measurable quantities are components of horizontal tensors evaluated at the observers trajectory on TM.

Here the tangent direction of the observer singles out the time and space components of the tensor field as usual, but the field components also depend on the observers direction as function.



Transformations between such observers turn out to be a groupoid based on the Lorentz group.

III Field Theories

The geometry of Finsler spacetime is built from tensors on TM; hence physical fields coupling to this geometry have to be of the same kind.

The construction of Lagrange densities on TM requires the canonical Sasaki-type metric

$$G=g^F_{ab}dx^a\otimes dx^b+\frac{g^F_{ab}}{F^2}\delta y^a\otimes\delta y^b\,.$$

To couple field theories to Finsler spacetime geometry we employ the following procedure:

Choose an action for a p-form T(x) on (M,g)

$$S[T,g]=\int_M[\sqrt{g}\,\mathcal{L}[T,dT,g]](x)\,,$$

use the Lagrangian for a zero homogenous p-form field T(x,y) on (TM,G), introduce Lagrange multipliers to restrict the p-form field to the horizontal space and finally integrate over the so called unit tangent bundle Σ={(x,y)∈TM | F(x,y)=1}

$$S_m[T,L]=\int_\Sigma[\sqrt{g^Fh^F}(\mathcal{L}[T,dT,L]+\lambda(1-P^H)T)](x,y)_{|\Sigma}.$$

Variation with respect to the field yields the equations of motion, variation with respect to the Lagrange multiplier ensures the vanishing of all non horizontal components on shell and variation with respect to the L function gives the source term of the gravitational dynamics

$$\mathcal{T}_{|\Sigma}=[\frac{rL}{\sqrt{g^Fh^F}}\frac{\delta S_m}{\delta L}]_\Sigma\,.$$

This coupling principle ensures that in case the Finsler spacetime is metric the field theories and the gravitational dynamics are identical to those from general relativity.

IV Gravity

The geodesic deviation equation on Finsler spacetimes gives rise to a tensor R causing relative gravitational acceleration

$$\nabla_{\dot{x}}\nabla_{\dot{x}}V^a=R^a{}_{bc}(x,\dot{x})\dot{x}^bV^c\,.$$

This non-linear curvature is built form the non-linear connection N through

$$R^a{}_{bc}=\delta_{[b}N^a{}_{c]}\,.$$

Without further derivatives, or other tensors depending on L, the natural curvature scalar is defined as

$$\mathcal{R}^{\mathcal{F}}=R^a{}_{ab}y^b\,.$$

The Finsler spacetime gravity action is

$$S[L,T]=\frac{c^4}{4\pi G}\int_\Sigma(\sqrt{g^Fh^F}\,\mathcal{R}^{\mathcal{F}})_{|\Sigma}+S_m[L,T]\,.$$

Variation with respect to L leads to the Finsler spacetime gravity field equation

$$[-g^{Fab}\bar{\partial}_a\bar{\partial}_b\mathcal{R}^{\mathcal{F}}+\frac{6}{F^2}\mathcal{R}^{\mathcal{F}}$$
$$-2g^{Fab}(\nabla_aS_b+S_aS_b+\bar{\partial}_a\nabla S_b)]_{|\Sigma}=\frac{4\pi G}{c^4}\mathcal{T}_{|\Sigma}\,.$$

It contains the curvature scalar, a measure of the departure from metric geometry S, and a Finsler version of the Levi-Civita derivative.

In case the function L is the metric length measure the Finsler gravity equation becomes equivalent to Einstein's equations

$$R_{ab}y^ay^b-\frac{1}{2}Rg_{ab}y^ay^b=\frac{8\pi G}{c^4}\mathcal{T}_{ab}y^ay^b\,.$$

Conclusion

We have construct a theory of gravity for spacetimes equipped with a general Finsler length measure.

In case the Finsler length equals the metric length our theory becomes general relativity, hence all solutions of the Einstein equations are solutions to our Finsler gravity equation. The implications of Finsler spacetime gravity on the dark universe can be studied by spherical symmetric and cosmological solutions that go beyond metric geometry.

A perturbative first order Finsler solution around the Schwarzschild and the Friedmann-Robertson-Walker metric is work in progress.

[1] D. N. Spergel et al. Astrophys. J. Suppl. 170, 377 (2007)
[2] Bao, Chern, Shen, An Introduction to Riemann-Finsler Geometry
[3] Pfeifer, Wohlfarth; Physical Review D 84, 044039 (2011)
[4] Pfeifer, Wohlfarth; Physical Review D 85, 064009 (2012)