Geodesic deviation in Kundt spacetimes of any dimension

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motivation

our goal was to suggest a method

of local identification and physical interpretation

of exact spacetimes in any dimension $D \geq 4$,

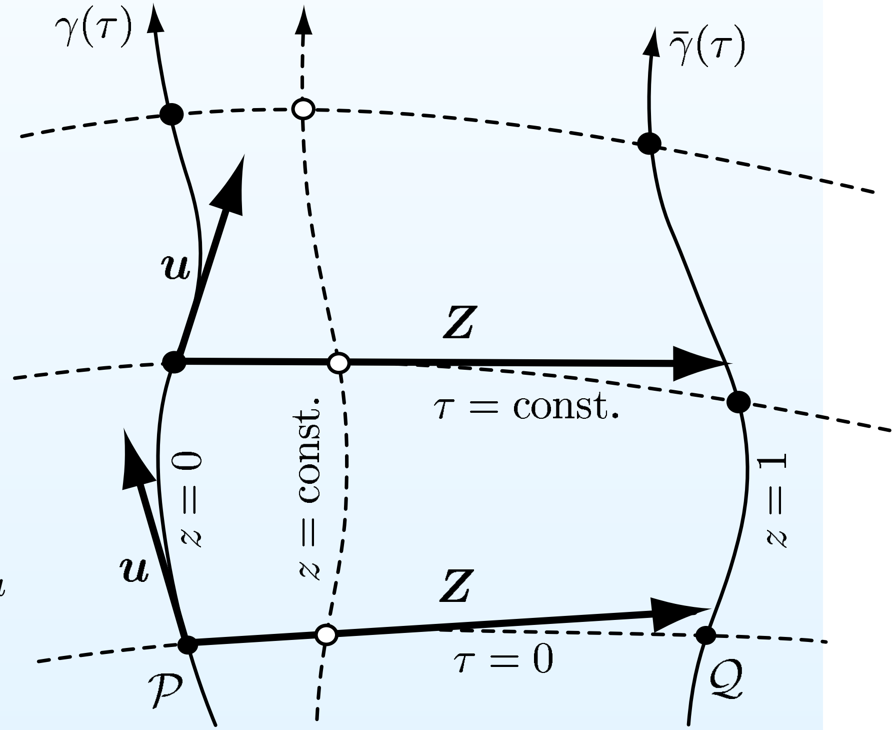
and apply it to the general Kundt family

equation of geodesic deviation

describes relative motion of free test particles without charge or spin in any dim D

Levi-Civita (1926), Synge (1926,1934), Pirani (1956), ...

$$\frac{\mathrm{D}^2 Z^\mu}{\mathrm{d}\,\tau^2} = R^\mu{}_{\alpha\beta\nu}\, u^\alpha u^\beta Z^\nu$$

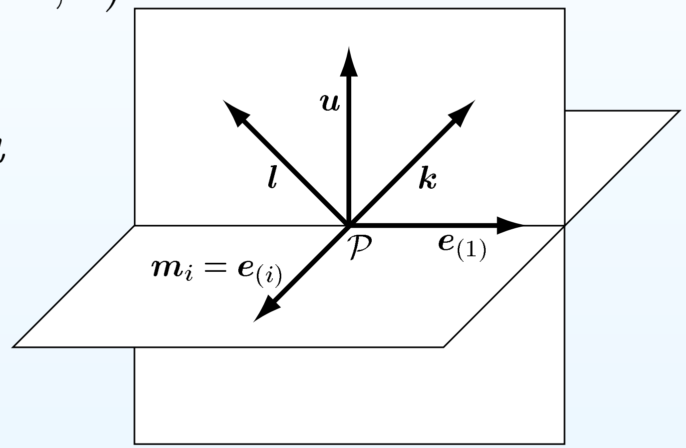


- $R^\mu{}_{\alpha\beta\nu}$ components of the Riemann curvature tensor
- u^α components of the velocity vector $\boldsymbol{u} = u^\alpha \partial_\alpha$ of the reference particle moving along a timelike geodesic $\gamma(\tau) = \{x^0(\tau), \ldots, x^{D-1}(\tau)\}$
- τ its proper time, so $u^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}$ and $\boldsymbol{u}\cdot\boldsymbol{u} = -1$
- Z^μ components of the separation vector $\boldsymbol{Z} = Z^\mu \partial_\mu$ connecting the reference particle with a nearby particle moving along a timelike geodesic $\bar{\gamma}(\tau)$

explicitly expresses the relative acceleration of nearby particles in terms of the local curvature and their actual relative position

invariant form of geodesic deviation

to obtain physical results independent of the choice of coordinates, it is natural to adopt Pirani's approach (1956) based on the use of components of all quantities with respect to an orthonormal frame $\{\boldsymbol{e}_a\}$: $\quad \boldsymbol{e}_a \cdot \boldsymbol{e}_b = \eta_{ab} \equiv \mathrm{diag}(-1, 1, \ldots, 1)$



$\boldsymbol{e}_{(0)} = \boldsymbol{u}$ velocity vector of the observer

$\boldsymbol{e}_{(\mathrm{i})}$ local spatial Cartesian basis orthogonal to $\boldsymbol{u}$

$\mathrm{i}=1,2,\ldots,D-1$

frame components Z^a of the separation vector $\boldsymbol{Z} = Z^a \boldsymbol{e}_a$ are then determined by $Z^{(0)} = 0$ and $Z^{(\mathrm{i})}(\tau)$ that solve

$$\ddot{Z}^{(\mathrm{i})} = R^{(\mathrm{i})}{}_{(0)(0)(\mathrm{j})}\, Z^{(\mathrm{j})}$$

$Z^{(\mathrm{i})} \equiv \boldsymbol{e}^{(\mathrm{i})} \cdot \boldsymbol{Z}$ actual relative spatial position of two test particles

$\ddot{Z}^{(\mathrm{i})} \equiv \boldsymbol{e}^{(\mathrm{i})} \cdot \frac{\mathrm{D}^2 \boldsymbol{Z}}{\mathrm{d}\,\tau^2}$ physical relative acceleration of these particles

$R_{(\mathrm{i})(0)(0)(\mathrm{j})} \equiv R_{\mu\alpha\beta\nu}\, e^{\mu}_{(\mathrm{i})} u^{\alpha} u^{\beta} e^{\nu}_{(\mathrm{j})}$ frame components of the Riemann tensor

canonical decomposition of the curvature tensor

- using the definition of the traceless Weyl tensor C_{abcd} we obtain

$$R_{(\mathrm{i})(0)(0)(\mathrm{j})} = C_{(\mathrm{i})(0)(0)(\mathrm{j})} + \tfrac{1}{D-2}\big(R_{(\mathrm{i})(\mathrm{j})} - \delta_{\mathrm{ij}}\, R_{(0)(0)}\big) - \tfrac{1}{(D-1)(D-2)}\, R\,\delta_{\mathrm{ij}}$$

- the Ricci tensor R_{ab} and Ricci scalar R can be expressed using Einstein's equations

$$R_{ab} - \tfrac{1}{2} R\, g_{ab} + \Lambda\, g_{ab} = 8\pi\, T_{ab} \quad \Rightarrow \quad \text{trace}\;\; R = \tfrac{2}{2-D}(8\pi\, T - D\,\Lambda)$$

Λ cosmological constant

T_{ab} energy-momentum tensor of matter fields

the invariant form of the equation of geodesic deviation thus becomes

$$\ddot{Z}^{(\mathrm{i})} = \frac{2}{(D-1)(D-2)}\,\Lambda\, Z^{(\mathrm{i})} + C_{(\mathrm{i})(0)(0)(\mathrm{j})}\, Z^{(\mathrm{j})} + \frac{8\pi}{D-2}\left[T_{(\mathrm{i})(\mathrm{j})}\, Z^{(\mathrm{j})} - \left(T_{(0)(0)} + \frac{2}{D-1}\,T\right) Z^{(\mathrm{i})}\right]$$

- finally, we analyze the orthonormal components of the "free gravitational field" C_{abcd} : these can be conveniently expressed using the Newman–Penrose-type scalars $\Psi_{A\,ij..}$ which are the components of the Weyl tensor with respect to an associated null frame

Newman–Penrose-type scalars $\Psi_{A\,ij..}$

orthonormal frame $\{\boldsymbol{u}, \boldsymbol{e}_{(1)}, \boldsymbol{e}_{(2)}, \boldsymbol{e}_{(3)}, \dots\}$ $\qquad \boldsymbol{e}_a \cdot \boldsymbol{e}_b = \eta_{ab} \equiv \text{diag}(-1, 1, \dots, 1)$

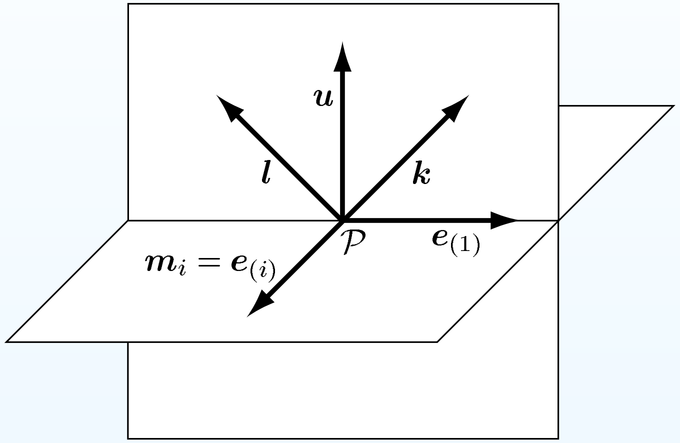
associated null frame $\{\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}_2, \boldsymbol{m}_3, \dots\}$

are simply related via

$$\boldsymbol{k} = \tfrac{1}{\sqrt{2}}(\boldsymbol{u} + \boldsymbol{e}_{(1)})$$

$$\boldsymbol{l} = \tfrac{1}{\sqrt{2}}(\boldsymbol{u} - \boldsymbol{e}_{(1)})$$

$$\boldsymbol{m}_i = \boldsymbol{e}_{(i)} \quad \text{for} \quad {\scriptstyle i=2,3,\dots,D-1}$$



$\boldsymbol{m}_i$ are $D-2$ transverse spatial vectors: $\boldsymbol{m}_i \cdot \boldsymbol{m}_j = \delta_{ij}$

$\boldsymbol{k}$ and $\boldsymbol{l}$ are future-oriented null vectors: $\quad \boldsymbol{k} \cdot \boldsymbol{k} = 0 = \boldsymbol{l} \cdot \boldsymbol{l}$ such that $\boldsymbol{k} \cdot \boldsymbol{l} = -1$

components of the Weyl tensor in this null frame are the Newman–Penrose scalars $\Psi_{A\,ij..}$

$$\begin{aligned}
\Psi_{0\,ij} &\equiv C_{abcd}\, k^a\, m_i^b\, k^c\, m_j^d \\
\Psi_{1\,ijk} &\equiv C_{abcd}\, k^a\, m_i^b\, m_j^c\, m_k^d \\
\Psi_{2\,ijkl} &\equiv C_{abcd}\, m_i^a\, m_j^b\, m_k^c\, m_l^d \\
\Psi_{2\,ij} &\equiv C_{abcd}\, k^a\, l^b\, m_i^c\, m_j^d \\
\Psi_{3\,ijk} &\equiv C_{abcd}\, l^a\, m_i^b\, m_j^c\, m_k^d \\
\Psi_{4\,ij} &\equiv C_{abcd}\, l^a\, m_i^b\, l^c\, m_j^d
\end{aligned}
\qquad
\begin{aligned}
\Psi_{1T^i} &\equiv C_{abcd}\, k^a\, l^b\, k^c\, m_i^d \\
\Psi_{2S} &\equiv C_{abcd}\, k^a\, l^b\, l^c\, k^d \\
\Psi_{2T^{ij}} &\equiv C_{abcd}\, k^a\, m_i^b\, l^c\, m_j^d \\
\Psi_{3T^i} &\equiv C_{abcd}\, l^a\, k^b\, l^c\, m_i^d
\end{aligned}$$

grouped by their boost weight

some properties of $\Psi_{A\,ij..}$

symmetries

$$\Psi_{0\,[ij]} = 0 \qquad \Psi_{0\,k}{}^{k} = 0$$
$$\Psi_{1\,i(jk)} = 0 \qquad \Psi_{1\,[ijk]} = 0$$
$$\Psi_{2\,ijkl} = \Psi_{2\,klij} \qquad \Psi_{2\,(ij)} = 0$$
$$\Psi_{2\,(ij)kl} = \Psi_{2\,ij(kl)} = \Psi_{2\,i[jkl]} = 0$$
$$\Psi_{3\,i(jk)} = 0 \qquad \Psi_{3\,[ijk]} = 0$$
$$\Psi_{4\,[ij]} = 0 \qquad \Psi_{4\,k}{}^{k} = 0$$

and tracing relations

$$\Psi_{1\,T^i} = \Psi_{1\,k}{}^{k}{}_{i}\,, \qquad \Psi_{3\,T^i} = \Psi_{3\,k}{}^{k}{}_{i}\,, \qquad \Psi_{2\,S} = \Psi_{2\,T^k}{}^{k} = \tfrac{1}{2}\Psi_{2\,kl}{}^{kl}$$
$$\Psi_{2\,T^{ij}} = \tfrac{1}{2}(\Psi_{2\,ikj}{}^{k} + \Psi_{2\,ij}) \qquad \Psi_{2T\,(ij)} = \tfrac{1}{2}\Psi_{2\,ikj}{}^{k}\,, \qquad \Psi_{2\,T[ij]} = \tfrac{1}{2}\Psi_{2\,ij}$$

the $C_{\mathrm{(i)(0)(0)(j)}}$ components of the Weyl tensor can now be expressed using $\Psi_{A\,ij..}$:

$$C_{(1)(0)(0)(1)} = \Psi_{2\,S}$$
$$C_{(1)(0)(0)(j)} = \frac{1}{\sqrt{2}}\left(\Psi_{1\,T^j} - \Psi_{3\,T^j}\right)$$
$$C_{(i)(0)(0)(j)} = -\frac{1}{2}\left(\Psi_{0\,ij} + \Psi_{4\,ij}\right) - \Psi_{2\,T^{(ij)}}$$

fully general & invariant form of geodesic deviation

$$\ddot{Z}^{(1)} = \frac{2}{(D-1)(D-2)}\,\Lambda\,Z^{(1)} + \Psi_{2\,S}\,Z^{(1)} + \frac{1}{\sqrt{2}}\big(\Psi_{1\,T^j} - \Psi_{3\,T^j}\big)\,Z^{(j)} + \frac{8\pi}{D-2}\Big[T_{(1)(1)}\,Z^{(1)} + T_{(1)(j)}\,Z^{(j)} - \Big(T_{(0)(0)} + \frac{2}{D-1}\,T\Big)\,Z^{(1)}\Big]$$

longitudinal spatial direction (1)

$$\ddot{Z}^{(i)} = \frac{2}{(D-1)(D-2)}\,\Lambda\,Z^{(i)} - \Psi_{2\,T^{(ij)}}\,Z^{(j)} + \frac{1}{\sqrt{2}}\big(\Psi_{1\,T^i} - \Psi_{3\,T^i}\big)\,Z^{(1)} - \frac{1}{2}\big(\Psi_{0\,ij} + \Psi_{4\,ij}\big)\,Z^{(j)} + \frac{8\pi}{D-2}\Big[T_{(i)(1)}\,Z^{(1)} + T_{(i)(j)}\,Z^{(j)} - \Big(T_{(0)(0)} + \frac{2}{D-1}\,T\Big)\,Z^{(i)}\Big]$$

transverse spatial directions $(i)=(2),(3),\ldots$

other equivalent notations for the NP Weyl scalars used in the literature:

$\Psi_{2\,S}$	$\Psi_{2\,T^{ij}}$	$\Psi_{1\,T^j}$	$\Psi_{3\,T^j}$	$\Psi_{0\,ij}$	$\Psi_{4\,ij}$	
$-C_{0101}$	$-C_{0i1j}$	$-C_{010j}$	C_{101j}	C_{0i0j}	C_{1i1j}	Coley *et al.* 2004, 2008
$-\Phi$	$-\Phi_{ij}$		Ψ_j		$2\,\Psi_{ij}$	Pravda *et al.* 2004, 2007
$-\Phi$	$-\Phi_{ij}$	$-\Psi_j$	Ψ'_j	Ω_{ij}	Ω'_{ij}	Durkee *et al.* 2010

canonical components of a gravitational field

and their specific effect on free test particles

vacuum case $T_{ab} = 0$ $\Rightarrow$ only contributions from Λ and the gravitational field that have a specific and unique character:

- Λ : isotropic influence of the background given by the cosmological constant

$$\begin{pmatrix} \ddot{Z}^{(1)} \\ \ddot{Z}^{(i)} \end{pmatrix} = \frac{2\,\Lambda}{(D-1)(D-2)} \begin{pmatrix} 1 & 0 \\ 0 & \delta_{ij} \end{pmatrix} \begin{pmatrix} Z^{(1)} \\ Z^{(j)} \end{pmatrix}$$

explicit solutions in parallelly propagated frames:

$$\Lambda = 0 : \quad Z^{(\mathrm{i})} = A_{\mathrm{i}}\,\tau + B_{\mathrm{i}}$$

$$\Lambda > 0 : \quad Z^{(\mathrm{i})} = A_{\mathrm{i}} \cosh\left[\sqrt{\frac{2\Lambda}{(D-1)(D-2)}}\,\tau\right] + B_{\mathrm{i}} \sinh\left[\sqrt{\frac{2\Lambda}{(D-1)(D-2)}}\,\tau\right]$$

$$\Lambda < 0 : \quad Z^{(\mathrm{i})} = A_{\mathrm{i}} \cos\left[\sqrt{\frac{2\,|\Lambda|}{(D-1)(D-2)}}\,\tau\right] + B_{\mathrm{i}} \sin\left[\sqrt{\frac{2\,|\Lambda|}{(D-1)(D-2)}}\,\tau\right]$$

typical relative motions of test particles in spacetimes of constant curvature
Minkowski, de Sitter, and anti-de Sitter space, respectively (Synge, 1934)

effect of canonical components on test particles

- $\Psi_{2\,S}$, $\Psi_{2\,T^{ij}}$: Newtonian components of the gravitational field

$$\begin{pmatrix} \ddot{Z}^{(1)} \\ \ddot{Z}^{(i)} \end{pmatrix} = \begin{pmatrix} \Psi_{2\,S} & 0 \\ 0 & -\Psi_{2\,T^{(ij)}} \end{pmatrix} \begin{pmatrix} Z^{(1)} \\ Z^{(j)} \end{pmatrix}$$

deformations that generalize classical Newtonian-type tidal effects in $D=4$ gravity

these terms are typically present in spacetimes of algebraic type D,
in particular around spherically symmetric static sources

the $(D-1)\times(D-1)$-dim matrix is symmetric and traceless since $\Psi_{2\,S} = \Psi_{2\,T^k}{}^k$

effect of canonical components on test particles

- $\Psi_{3\,T^j}$,$\Psi_{1\,T^j}$: longitudinal components of the gravitational field

$$\begin{pmatrix} \ddot{Z}^{(1)} \\ \ddot{Z}^{(i)} \end{pmatrix} = -\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \Psi_{3\,T^j} \\ \Psi_{3\,T^i} & 0 \end{pmatrix} \begin{pmatrix} Z^{(1)} \\ Z^{(j)} \end{pmatrix}$$

cause longitudinal deformations of a cloud of test particles

typical for spacetimes of algebraic type III

the $(D-2)$ scalars Ψ_{3T^i} combine motion in the privileged spatial direction $+\boldsymbol{e}_{(1)}$ with motion in the transverse directions $\boldsymbol{e}_{(i)}$

the $(D-2)$ scalars Ψ_{1T^i} combine motion in the privileged spatial direction $-\boldsymbol{e}_{(1)}$ with motion in the transverse directions $\boldsymbol{e}_{(i)}$

$\Psi_{1\,T^i}$ are equivalent to $\Psi_{3\,T^i}$ under the interchange $\boldsymbol{k} \leftrightarrow \boldsymbol{l}$, but $\boldsymbol{k} \cdot \boldsymbol{e}_{(1)} > 0$ while $\boldsymbol{l} \cdot \boldsymbol{e}_{(1)} < 0$

effect of canonical components on test particles

- $\Psi_{4\,ij}$,$\Psi_{0\,ij}$: transverse gravitational waves propagating in the directions $\pm\boldsymbol{e}_{(1)}$

$$\begin{pmatrix} \ddot{Z}^{(1)} \\ \ddot{Z}^{(i)} \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & \Psi_{4\,ij} \end{pmatrix} \begin{pmatrix} Z^{(1)} \\ Z^{(j)} \end{pmatrix}$$

cause purely transverse effects on a set of test particles: no acceleration in the direction $\boldsymbol{e}_{(1)}$

typical for spacetimes of algebraic type N

the symmetric $(\Psi_{4\,ij} = \Psi_{4\,ji})$, traceless $(\Psi_{4\,k}{}^{k} = 0)$ matrix of dimension $(D-2)\times(D-2)$ encodes $\frac{1}{2}D(D-3)$ independent polarization modes propagating along the null direction $\boldsymbol{k}$, i.e., the spatial direction $+\boldsymbol{e}_{(1)}$

complementarily, $\Psi_{0\,ij}$ represents gravitational waves propagating along $\boldsymbol{l}$, i.e. $-\boldsymbol{e}_{(1)}$

$\Psi_{4\,ji}$ are equivalent to $\Psi_{0\,ji}$ under the interchange $\boldsymbol{k} \leftrightarrow \boldsymbol{l}$, and $\boldsymbol{k}\cdot\boldsymbol{e}_{(1)} > 0$ while $\boldsymbol{l}\cdot\boldsymbol{e}_{(1)} < 0$

Kundt spacetimes

Kundt (1961, 1962) $\star\star\star$ 50th anniversary $\star\star\star$

introduced and studied all four-dimensional geometries that admit a geodesic null congruence generated by $\boldsymbol{k}$ that is

- twist-free: $0 = \text{Tr}\, A^2 \equiv -k_{[\alpha;\beta]}\, k^{\alpha;\beta}$
- shear-free: $0 = \text{Tr}\, \sigma^2 \equiv k_{(\alpha;\beta)}\, k^{\alpha;\beta} - \frac{1}{D-2}(k^{\alpha}{}_{;\alpha})^2$
- non-expanding: $0 = \theta \equiv \frac{1}{D-2} k^{\alpha}{}_{;\alpha}$

such metrics in any dimension D can be written as

$$\mathrm{d}s^2 = g_{ij}(x,u)\,\mathrm{d}x^i\,\mathrm{d}x^j + 2\,g_{ui}(x,u,r)\,\mathrm{d}x^i\,\mathrm{d}u - 2\,\mathrm{d}u\,\mathrm{d}r + g_{uu}(x,u,r)\,\mathrm{d}u^2$$

- $x \equiv (x^i) \equiv (x^1, x^2, \ldots, x^{D-2})$: spatial coordinates on a transverse $(D-2)$-dim manifold
- $u =$ const: null hypersurfaces to which $\boldsymbol{k}$ is normal
- r: affine parameter along the geodesics generated by $\boldsymbol{k} = \partial_r$

important members of the Kundt family

- pp-waves (CCNV spacetimes)
  defined geometrically as admitting a covariantly constant null vector field $\boldsymbol{k}$
  necessarily independent of r: (Brinkmann, 1925)

$$\mathrm{d}s^2 = g_{ij}\ \mathrm{d}x^i\,\mathrm{d}x^j + 2\,e_i\ \mathrm{d}x^i\,\mathrm{d}u - 2\ \mathrm{d}u\ \mathrm{d}r + c\ \mathrm{d}u^2$$

- VSI spacetimes
  scalar curvature invariants of all orders vanish
  transverse space is flat, $g_{ij} = \delta_{ij}$: (Coley et al., 2006 etc)

$$\mathrm{d}s^2 = \delta_{ij}\ \mathrm{d}x^i\,\mathrm{d}x^j + 2\,(e_i + f_i\,r)\ \mathrm{d}x^i\,\mathrm{d}u - 2\ \mathrm{d}u\ \mathrm{d}r + (a\,r^2 + b\,r + c)\,\mathrm{d}u^2$$

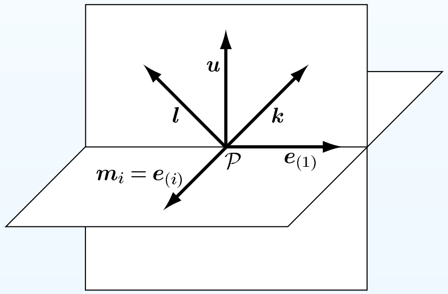
- gyratons
  field of a localized spinning source that propagates at the speed of light
  the simplest gyraton with $\Lambda = 0$ has the metric: (Bonnor, 1970, Frolov et al., 2005)

$$\mathrm{d}s^2 = \delta_{ij}\ \mathrm{d}x^i\,\mathrm{d}x^j + 2\,e_i\ \mathrm{d}x^i\,\mathrm{d}u - 2\ \mathrm{d}u\ \mathrm{d}r + c\ \mathrm{d}u^2$$

geodesic deviation in Kundt spacetimes

for the metric $$\mathrm{d}s^2 = g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j + 2\,g_{ui}\,\mathrm{d}x^i\,\mathrm{d}u - 2\,\mathrm{d}u\,\mathrm{d}r + g_{uu}\,\mathrm{d}u^2$$

the interpretation null frame adapted to a general observer that has the velocity $\boldsymbol{u} = \dot{r}\,\partial_r + \dot{u}\,\partial_u + \dot{x}^2\,\partial_{x^2} + \ldots + \dot{x}^{D-1}\partial_{x^{D-1}}$ is



$$\boldsymbol{k} = \frac{1}{\sqrt{2}\,\dot{u}}\,\partial_r$$

$$\boldsymbol{l} = \left(\sqrt{2}\,\dot{r} - \frac{1}{\sqrt{2}\,\dot{u}}\right)\partial_r + \sqrt{2}\,\dot{u}\,\partial_u + \sqrt{2}\,\dot{x}^2\,\partial_{x^2} + \ldots + \sqrt{2}\,\dot{x}^{D-1}\partial_{x^{D-1}}$$

$$\boldsymbol{m}_i = \frac{1}{\dot{u}}(g_{uk}\dot{u} + g_{jk}\,\dot{x}^j)\,m_i^k\,\partial_r + m_i^2\,\partial_{x^2} + \ldots + m_i^{D-1}\partial_{x^{D-1}}$$

where $g_{kl}\,m_i^k\,m_j^l = \delta_{ij}$

Weyl tensor projected on this null frame gives the following nonvanishing scalars:

(after a somewhat lengthy calculation)

geodesic deviation in a general Kundt spacetime

$$\ddot{Z}^{(1)} = \frac{2\,\Lambda}{(D-1)(D-2)}\,Z^{(1)} + \Psi_{2\,S}\,Z^{(1)} - \frac{1}{\sqrt{2}}\,\Psi_{3\,T^j}\,Z^{(j)}$$

$$\ddot{Z}^{(i)} = \frac{2\,\Lambda}{(D-1)(D-2)}\,Z^{(i)} - \Psi_{2\,T^{ij}}\,Z^{(j)} - \frac{1}{\sqrt{2}}\,\Psi_{3\,T^i}\,Z^{(1)} - \frac{1}{2}\,\Psi_{4\,ij}\,Z^{(j)}$$

longitudinal spatial direction

transverse spatial directions $(i)=(2),(3),\dots$

plus contributions from the matter T_{ab}

$$\Psi_{2S} = -C_{ruru}$$

$$\Psi_{2T^{ij}} = C_{rkul}\,m_i^k m_j^l$$

$$\Psi_{3T^i} = \sqrt{2}\Big[(C_{ruru}\,g_{mk} - C_{rmuk} - C_{rumk})\,\dot{x}^m + (C_{ruru}\,g_{uk} - C_{ruuk})\,\dot{u}\Big]\,m_i^k$$

$$\begin{aligned}\Psi_{4ij} = 2\Big[&\Big(C_{ruru}\,g_{km}g_{ln} + C_{rkul}\,g_{mn} - 2(C_{rnul} + C_{runl})g_{mk} + C_{mknl}\Big)\,\dot{x}^m\dot{x}^n \\ &+2\Big(C_{ruru}\,g_{uk}g_{ln} + C_{rkul}\,g_{mu} - (C_{rmul} + C_{ruml})g_{uk} - C_{ruuk}\,g_{lm} + C_{ukml}\Big)\,\dot{x}^m\dot{u} \\ &+\Big(C_{ruru}\,g_{uk}g_{ul} + C_{rkul}\,g_{uu} - 2C_{ruuk}\,g_{ul} + C_{ukul}\Big)\,\dot{u}^2\Big]\,m_{(i}^k m_{j)}^l\end{aligned}$$

these scalars combine the specific curvature with kinematics

Weyl tensor C_{abcd} $\qquad$ $\dot{x}^m, \dot{u}$ velocity components of the observer

no particular field equations have so far been imposed !

important subcase: vacuum pp-waves

for vacuum metrics $$\boxed{\mathrm{d}s^2 = g_{ij}\ \mathrm{d}x^i\,\mathrm{d}x^j + 2\,e_i\ \mathrm{d}x^i\,\mathrm{d}u - 2\ \mathrm{d}u\ \mathrm{d}r + c\ \mathrm{d}u^2}$$

we necessarily have $\Lambda = 0$, $R_{ab} = 0 \Rightarrow \Psi_{2S} = 0 = \Psi_{2T^{ij}}\,,\ \Psi_{3T^i} = 0$

$$\ddot{Z}^{(1)} = 0$$
$$\ddot{Z}^{(i)} = -\tfrac{1}{2}\,\Psi_{4\,ij}\,Z^{(j)}$$

longitudinal spatial direction (1)

transverse spatial directions $(i) = (2), (3), \ldots$

purely transverse gravitational waves propagating in the spatial direction $\boldsymbol{e}_{(1)}$

$$\boxed{\Psi_{4^{ij}} = 2\left({}^{\mathrm{s}}R_{kmln}\,\dot{x}^m\dot{x}^n + 2R_{kmlu}\,\dot{x}^m\dot{u} + R_{kulu}\,\dot{u}^2\right) m^k_{(i}m^l_{j)}}$$

$$R_{ijkl} = {}^{\mathrm{s}}R_{ijkl}$$

$$R_{uijk} = \tfrac{1}{2}(e_{k,ij} - e_{j,ik} + g_{ij,uk} - g_{ik,uj}) + {}^{\mathrm{s}}\Gamma^m_{ij}\left(\tfrac{1}{2}g_{km,u} + e_{[m,k]}\right) - {}^{\mathrm{s}}\Gamma^m_{ik}\left(\tfrac{1}{2}g_{jm,u} + e_{[m,j]}\right)$$

$$R_{iuju} = \tfrac{1}{2}(e_{i,uj} + e_{j,ui} - c_{,ij} - g_{ij,uu}) + g^{kl}\left(\tfrac{1}{2}g_{ik,u} + e_{[k,i]}\right)\left(\tfrac{1}{2}g_{jl,u} + e_{[l,j]}\right) - {}^{\mathrm{s}}\Gamma^k_{ij}\left(e_{k,u} - \tfrac{1}{2}c_{,k}\right)$$

properties of the simplest vacuum pp-waves

VSI spacetimes ($g_{ij} = \delta_{ij}$ for $i, j = 2, 3, \ldots, D-1$) without gyratons ($e_i = 0$)

$$\mathrm{d}s^2 = \delta_{ij}\ \mathrm{d}x^i\,\mathrm{d}x^j - 2\ \mathrm{d}u\ \mathrm{d}r + c\,(x^i, u)\ \mathrm{d}u^2$$

in the parallelly propagated interpretation frame

$$\boldsymbol{k} = \frac{1}{\sqrt{2}\,\dot{u}}\,\partial_r, \qquad \boldsymbol{l} = \sqrt{2}\,\boldsymbol{u} - \boldsymbol{k}, \qquad \boldsymbol{m}_i = \frac{\dot{x}^i}{\dot{u}}\,\partial_r + \partial_{x^i} \qquad \dot{u} = \text{const.}$$

geodesic deviation takes the form:

$$\ddot{Z}^{(1)} = 0$$
$$\ddot{Z}^{(i)} = \tfrac{1}{2}\,\dot{u}^2\,c_{,ij}\,Z^{(j)}$$

wave amplitudes are directly given by $\dfrac{\partial^2 c}{\partial x^i\,\partial x^j}$

$(D-2)\times(D-2)$ matrix $\Psi_{4^{ij}} = -\dot{u}^2\,c_{,ij}$ is symmetric and traceless because vacuum Einstein equations read $\Delta\,c \equiv \delta^{ij}\,c_{,ij} = 0 \qquad \Rightarrow$

gravitational waves are transverse and have $\frac{1}{2}D(D-3)$ independent polarization modes represented by the free components of the amplitude matrix $\Psi_{4^{ij}}$

homogeneous gravitational waves in any dimension D

let us assume that the function c in the metric $\mathrm{d}s^2 = \delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j - 2\,\mathrm{d}u\,\mathrm{d}r + c\,\mathrm{d}u^2$ is a quadratic form of the spatial coordinates:

$c = \sum_{i=2}^{D-1} \mathcal{A}_i\,(x^i)^2$ where $\mathcal{A}_i$ are constants satisfying $\sum_{i=2}^{D-1} \mathcal{A}_i = 0$

wave amplitudes are then given by the diagonal traceless matrix

$$\Psi_{4^{ij}} = -2\,\dot{u}^2 \begin{pmatrix} \mathcal{A}_2 & 0 & 0 & \cdots \\ 0 & \mathcal{A}_3 & 0 & \cdots \\ 0 & 0 & \mathcal{A}_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

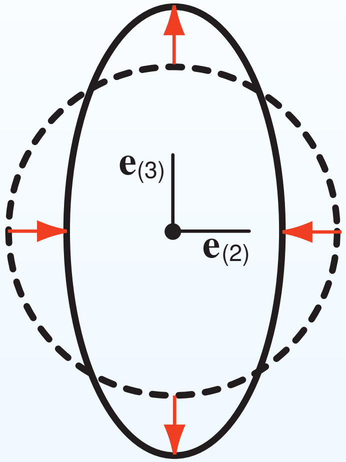
relative motion of test particles (initially at rest) can be explicitly integrated to

$$Z^{(i)}(\tau) = \begin{cases} Z_0^{(i)}\,\cosh\left(\sqrt{\mathcal{A}_i}\,|\dot{u}|\,\tau\right) & \text{for } \mathcal{A}_i > 0 \quad \text{particles recede} \\ Z_0^{(i)}\,\cos\left(\sqrt{-\mathcal{A}_i}\,|\dot{u}|\,\tau\right) & \text{for } \mathcal{A}_i < 0 \quad \text{particles approach} \\ Z_0^{(i)} & \text{for } \mathcal{A}_i = 0 \quad \text{no influence} \end{cases}$$

new observable effects due to higher dimensions

assume a gravitational wave propagating in the direction $\boldsymbol{e}_{(1)}$ of a D-dim spacetime

in the transverse $(D-2)$-dim subspace, we observe:



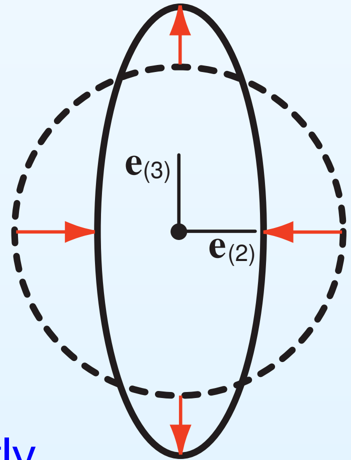
$D=4$ classical general relativity

eigenvalues of the matrix are $\mathcal{A}_2, \mathcal{A}_3$

$\mathcal{A}_2 = -\mathcal{A}_3$ traceless property

$$\Psi_{4^{ij}} = -2\,\dot{u}^2 \begin{pmatrix} -\mathcal{A}_3 & 0 \\ 0 & \mathcal{A}_3 \end{pmatrix}$$

simultaneously (non)trivial $\begin{cases} \mathcal{A}_3 \neq 0 & \text{wave} \\ \mathcal{A}_3 = 0 & \text{NO wave} \end{cases}$



$D=5$ higher-dimensional gravity

eigenvalues are $\mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$

$\mathcal{A}_2 = -(\mathcal{A}_3 + \mathcal{A}_4)$

$$\Psi_{4^{ij}} = -2\,\dot{u}^2 \begin{pmatrix} -(\mathcal{A}_3+\mathcal{A}_4) & 0 & 0 \\ 0 & \mathcal{A}_3 & 0 \\ 0 & 0 & \mathcal{A}_4 \end{pmatrix}$$

$\uparrow$ observable by detectors as a VIOLATION of the TT-property in our $(1+3)$-dim universe

$\uparrow$ not directly observable by our detectors

$$\mathcal{A}_4 = -(\mathcal{A}_3 + \mathcal{A}_2)$$

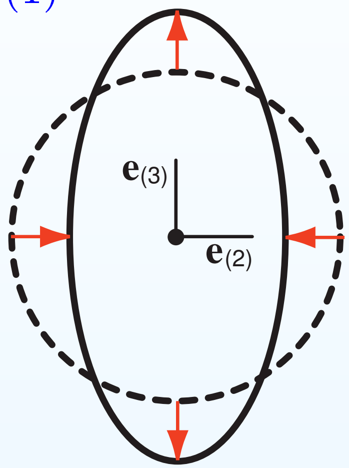
further special effects due to higher dimensions

particular subcases of gravitational waves propagating in the direction $\boldsymbol{e}_{(1)}$
in $D=5$ spacetime:

- $\boxed{\mathcal{A}_4 = 0}$

$$\Psi_{4^{ij}} = -2\,\dot{u}^2 \begin{pmatrix} -\mathcal{A}_3 & 0 & 0 \\ 0 & \mathcal{A}_3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

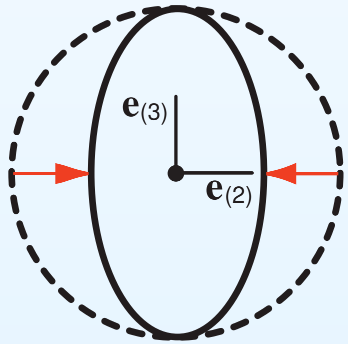
mimics standard GR



- $\boxed{\mathcal{A}_3 = 0}$

$$\Psi_{4^{ij}} = -2\,\dot{u}^2 \begin{pmatrix} -\mathcal{A}_4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathcal{A}_4 \end{pmatrix}$$

anomalous behaviour



also, the gravitational wave can propagate in the extra spatial dimension $\boldsymbol{e}_{(4)}$
due to the swap $\boldsymbol{e}_{(1)} \leftrightarrow \boldsymbol{e}_{(4)}$, this implies $\mathcal{A}_4 = 0$ and $\mathcal{A}_1 \neq 0$
we would observe an anomalous longitudinal deformation of a cloud of test particles

summary of the main results

- we explicitly derived the general equation of geodesic deviation in a natural reference frame adapted to any observer
- this can be a useful tool illuminating specific local effects of the gravitational field in an arbitrary dimension D
- canonical decomposition of relative accelerations of test particles consists of:
    - isotropic influence of the cosmological constant Λ — conflat
    - Newtonian-like tidal components Ψ_{2S}, $\Psi_{2T^{ij}}$ — type D
    - longitudinal effects Ψ_{3T^j} and Ψ_{1T^j} — type III
    - transverse gravitational waves Ψ_{4ij} and Ψ_{0ij} — type N
- gravitational waves propagating in the spatial direction $\boldsymbol{e}_{(1)}$ are described by Ψ_{4ij}, which is a symmetric and traceless matrix of dimension $(D-2)\times(D-2)$
- this encodes $\frac{1}{2}D(D-3)$ independent polarization modes
- explicit important example: Kundt class of spacetimes (including pp-waves, VSI, gyratons)
- due to the coupling between the eigenvalues of Ψ_{4ij}, gravitational waves in higher dimensions could be observed in our 4-dim world as a violation of TT

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