On a five-dimensional version of the Goldberg-Sachs theorem*

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- Algebraic classification of the Weyl tensor in arbitrary dimension
- NP and GHP formalism in higher dimensions
- Goldberg-Sachs theorem
  - Goldberg-Sachs theorem in 4D
  - HD "geodesic" part
  - HD "shearfree" part
    - type N, III
    - Goldberg-Sachs theorem in 5D
    - Examples
    - Optical constraint
- Conclusions

Let us work in the **frame**

$$\boldsymbol{m}_{(0)} = \boldsymbol{\ell},\ \boldsymbol{m}_{(1)} = \boldsymbol{n},\ \boldsymbol{m}_{(i)}, \quad i, j, k = 2 \ldots n-1,$$

with two *null* vectors $\boldsymbol{n}$, $\boldsymbol{\ell}$

$$\ell^a \ell_a = n^a n_a = 0, \quad \ell^a n_a = 1, \quad a = 0 \ldots n-1$$

and $n-2$ *spacelike* vectors

$$\boldsymbol{m}_{(i)}, \quad m^a_{(i)} m_{(j)a} = \delta_{ij}, \qquad i, j, k = 2 \ldots n-1.$$

metric

$$g_{ab} = 2\ell_{(a} n_{b)} + \delta_{ij} m_a^{(i)} m_b^{(j)}.$$

Classification of the Weyl tensor [Coley, Milson, Pravda, P, CQG, 2004]

* sorting Weyl components according to their *boost weight* b

- under boost $\hat{\boldsymbol{\ell}} = \lambda \boldsymbol{\ell},\ \hat{\boldsymbol{n}} = \lambda^{-1} \boldsymbol{n},\ \hat{\boldsymbol{m}}^{(i)} = \boldsymbol{m}^{(i)}$: $\hat{q} = \lambda^b q$ $\big(b(\boldsymbol{\ell}) = 1 \big)$

* according to (non)existence of *Weyl aligned null directions (WANDs)* (generalization of PNDs) and their multiplicity

* in 4D equivalent to the Petrov classification

$$\boldsymbol{C} = (\boldsymbol{C})_{(+2)} + (\boldsymbol{C})_{(+1)} + (\boldsymbol{C})_{(0)} + (\boldsymbol{C})_{(-1)} + (\boldsymbol{C})_{(-2)}$$

alg. t.	G	I	II/D	III	N
align. t.	(0,0)	(1,0)	(2,0)/(2,2)	(3,0)	(4,0)
		$\longmapsto \boldsymbol{\ell}$ WAND	$\longmapsto \boldsymbol{\ell}$ *multiple*	*algebraically*	WAND *special*
GHP compts.	Ω_{ij}	Ψ_{ijk}, Ψ_i	Φ_{ijkl}, Φ_{ij}, $2\Phi^{A}_{ij}, \Phi$	Ψ'_{ijk}, Ψ'_i	Ω'_{ij}

- Myers-Perry black hole (Kerr): type D in arbitrary dimension.

- Kerr-Schild spacetimes in arbitrary dimension

$$g_{ab} = \Omega\eta_{ab} - 2\mathcal{H}k_a k_b,$$

  with geodetic k_a - type II or more special, includes Myers-Perry black holes and type N pp-waves.

- All VSI spacetimes (spacetimes with vanishing curvature invariants) have the Weyl and Ricci tensors of type III or more special.

- *Ricci rotation coefficients* L_{ab}, N_{ab} and $\overset{i}{M}_{ab}$ are defined by

$$\ell_{a;b} = L_{cd} m_a^{(c)} m_b^{(d)}\ , \quad n_{a;b} = N_{cd} m_a^{(c)} m_b^{(d)}\ , \quad m_{a;b}^{(i)} = \overset{i}{M}_{cd}\, m_a^{(c)} m_b^{(d)}$$

e.g $\ell_{a;b} = L_{11}\ell_a\ell_b + L_{10}\ell_a n_b + L_{1i}\ell_a m_b^{(i)} + L_{i1} m_a^{(i)}\ell_b + L_{i0} m_a^{(i)} n_b + L_{ij} m_a^{(i)} m_b^{(j)}$

- ℓ is *geodetic* iff $L_{i0} = \kappa_i = 0$
- for geodetic ℓ, *affinely parametrized* $L_{10} = 0$

$$L_{ij} = \rho_{ij} = \sigma_{ij} + \theta\delta_{ij} + A_{ij},$$

shear $\quad \sigma_{ij} \equiv \rho_{(ij)} - \frac{1}{n-2}\rho_{kk}\delta_{ij},\ \sigma^2 \equiv \sigma_{ij}\sigma_{ji} = \ell_{(a;b)}\ell^{a;b} - \frac{1}{n-2}\left(\ell^a_{\ ;a}\right)^2,$

expansion $\quad \theta \equiv \frac{1}{n-2}\rho_{kk} = \frac{1}{n-2}\ell^a_{\ ;a}$

twist $\quad A_{ij} \equiv \rho_{[ij]},\ \omega^2 \equiv -A_{ij}A_{ji} = \ell_{[a;b]}\ell^{a;b}$

- *Ricci identities* - $v_{a;bc} - v_{a;cb} = R_{sabc}v^s$ projections on the frame ($v^a = \ell^a$, n^a and $m^a_{(i)}$) - set of first order PDEs, e.g. *Sachs* eq. for type I (or more special) with $R_{00} = 0$ (e.g. vacuum) and frame parallely propagated along a geodetic congruence $D\rho_{ij} = -\rho_{ik}\rho_{kj}$, $\quad D \equiv \ell^a\nabla_a$.
  [NP: M.Ortaggio, V. Pravda, A.P., CQG,2007; GHP: M. Durkee, V.P., A.P., H.S. Reall, CQG, 2010]
- *Bianchi identities* - projections of $R_{ab\{cd;e\}} = R_{abcd;e} + R_{abde;c} + R_{abec;d} = 0$
  - set of first order PDEs. [NP: V.P., A.P., A. Coley, R. Milson, CQG, 2004; GHP]

Goldberg-Sachs theorem (in 4D):
In an Einstein spacetime (i.e. $R_{ab} = (R/d)g_{ab}$), which is not conformally flat, a null vector field is a repeated principal null direction (of the Weyl tensor) if, and only if, it is geodesic and shear-free.

$\Psi_0 = 0 = \Psi_1 \quad \Leftrightarrow \quad \kappa = 0 = \sigma$, i.e.

$\Omega_{ij} = 0 = \psi_i \quad \Leftrightarrow \quad \kappa_i = 0,\ \rho_{33} = \rho_{22},\ \rho_{32} = -\rho_{23}$

$$\rho_{ij} = b \begin{pmatrix} 1 & a \\ -a & 1 \end{pmatrix}$$

Examples:

- $a \neq 0$: Kerr
- $a = 0$: Schwarzschild
- $a = 0 = b$: Kundt

Goldberg-Sachs theorem - HD “geodetic” part

- in HD there exist vacuum spacetimes with *non-geodetic* multiple WANDs

[Pravda, Pravdová, Ortaggio, CQG, 2007]

However:

Theorem - “geodetic” part of the Goldberg-Sachs theorem

[Durkee, Reall, CQG, 2009] : *A vacuum spacetime admitting a non-geodetic multiple WAND always also admits another multiple geodetic WAND.*

Proposition:

An Einstein spacetime of

- type N

- type III non-twisting

- type III in 5D

- 'generic' type III

have the optical matrix of the form

$$\rho_{ij} = \begin{pmatrix} b & a & 0 & \dots \\ -a & b & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Note that shear of the only non-vanishing two-block in ρ_{ij} is zero and ρ_{ij} satisfies *optical constraint*

$$\rho_{ik}\rho_{jk} \propto S_{ij}.$$

This in fact holds in much more general context.

- Algebraic Bianchi eqs.:

$$2\Phi^{A}_{[jk|}\rho_{i|l]} - 2\Phi_{i[j}\rho_{kl]} + \Phi_{im[jk|}\rho_{m|l]} = 0$$

$$2\Phi_{[kj]}\rho_{ij} + \Phi_{ij}\rho_{kj} - \Phi_{ji}\rho_{jk} + 2\Phi_{ij}\rho_{jk} - \Phi_{ik}\rho + \Phi\rho_{ik} + \Phi_{ijkl}\rho_{jl} = 0$$

symmetric and antisymmetric parts of the second one read

$$\left(2\Phi_{kj} - \Phi_{jk}\right)S_{ij} + \left(2\Phi_{ij} - \Phi_{ji}\right)S_{jk} - \Phi^{S}_{ik}\rho + \Phi S_{ik} + \Phi_{ijkl}S_{jl} = 0 \quad (1)$$

$$\Phi_{jk}A_{ji} + \Phi_{ji}A_{kj} + \Phi_{ij}\rho_{jk} - \Phi_{kj}\rho_{ji} + \Phi^{A}_{ki}\rho + \Phi A_{ik} + \Phi_{ijkl}A_{jl} = 0 \quad (2)$$

- Differential Bianchi eqs.:

$$\begin{aligned} \text{þ}\Phi_{ij} &= -(\Phi_{ik} + 2\Phi^{A}_{ik} + \Phi\delta_{ik})\rho_{kj} \\ -\text{þ}\Phi_{ijkl} &= 4\Phi^{A}_{ij}\rho_{[kl]} - 2\Phi_{[k|i}\rho_{j|l]} + 2\Phi_{[k|j}\rho_{i|l]} + 2\Phi_{ij[k|m}\rho_{m|l]} \end{aligned}$$

- Ricci (Sachs) eq.: $\text{þ}\rho_{ij} = -\rho_{ik}\rho_{kj}$
- New algebraic eq. obtained by differentiating (1)

$$\left(2\Phi_{kj} - \Phi_{jk}\right)\rho_{il}\rho_{jl} + \left(2\Phi_{ij} - \Phi_{ji}\right)\rho_{jl}\rho_{kl} - \Phi^{S}_{ik}\rho_{jl}\rho_{jl} + \Phi\rho_{il}\rho_{kl} + \Phi_{ijkl}\rho_{js}\rho_{ls} = 0.$$

GS theorem in 5D:

In a 5d algebraically special Einstein spacetime that is not conformally flat, there exists a geodesic multiple WAND $\boldsymbol{\ell}$ *and one can choose the orthonormal basis vectors* $\boldsymbol{m}^{(i)}$ *so that the optical matrix of* $\boldsymbol{\ell}$ *takes one of the forms*

$$i) \qquad b \begin{pmatrix} 1 & a & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1+a^2 \end{pmatrix},$$

$$ii) \qquad b \begin{pmatrix} 1 & a & 0 \\ -a & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$iii) \qquad b \begin{pmatrix} 1 & a & 0 \\ -a & -a^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

If the spacetime is type III or type N then the form must be $ii)$.

* 6 parametres $\rightarrow$ 2 parameters

- *Case i)* (rank 3 optical matrix)
  - non-twisting $a = 0$
    - Robinson-Trautmann class - Schwarzschild-Tangherlini BH
  - twisting $a \neq 0$
    - Myers-Perry black hole
    - all non-degenerate (i.e. $\det \rho \neq 0$) Einstein Kerr-Schild metrics with Minkowski or (A)dS background
    - 5d Kaluza-Klein bubble obtained by analytic continuation of a singly spinning Myers-Perry solution

- *Case ii)* (rank 2 optical matrix)
  - non-twisting $a = 0$
    - a direct or warped product of any 4d Einstein type II Robinson-Trautmann metric (e.g. the Schwarzschild black string solution)
  - twisting $a \neq 0$
    - rich class of Ricci-flat solutions - product of a 4d Ricci-flat algebraically special solution with a flat 5th direction, e.g. Kerr black string,
    - examples with non-vanishing cosmological constant - warped product of a 4d algebraically special Einstein spacetime with a 5th direction

- *Case iii)* (rank 1 optical matrix)
  - non-twisting $a = 0$
    - ∗ non-twisting, expanding and shearing geodesic multiple WAND in $dS_3 \times S^2$
    - ∗ the expanding Kaluza-Klein bubble solution (analytically continued 5d Schwarzschild) (type D), $\Phi_{ij} = \mathrm{diag}(-\Phi, \Phi, \Phi)$
  - twisting $a \neq 0$
    - ∗ genuine type II ?
    - ∗ all type D spacetimes - known = algebraically special metrics admitting a non-geodesic multiple WANDs - two subfamilies:

      1) direct products $dS_3 \times S^2$ and $AdS_3 \times H^2$

      2) analytical continuation of the 5d Schwarzschild solution (e.g. the Kaluza-Klein bubble) - generalized to include a cosmological constant Λ and planar or hyperbolic symmetry

* Cases i), ii), and iii) for $a = 0$ satisfy *optical constraint* $\rho_{ik}\rho_{jk} \propto S_{ij}$.

Optical constraint holds e.g. for
* Ricci-flat [Ortaggio, Pravda, P., CQG, 2009] and Einstein [Málek, Pravda, CQG, 2011] Kerr-Schild spacetimes (e.g. the Myers-Perry black hole),
* all asymptotically flat type II spacetimes admitting a non-degenerate ($\det \boldsymbol{\rho} \neq 0$) geodesic multiple WAND,
* all Einstein spacetimes of type N,
* Einstein spacetimes of type III with additional assumptions.

OC: block-diagonal form

$$\boldsymbol{\rho} = \alpha \ \mathrm{diag}\left(1,\ldots,1,\frac{1}{1+\alpha^2 b_1^2}\begin{bmatrix} 1 & -\alpha b_1 \\ \alpha b_1 & 1 \end{bmatrix},\ldots,\frac{1}{1+\alpha^2 b_\nu^2}\begin{bmatrix} 1 & -\alpha b_\nu \\ \alpha b_\nu & 1 \end{bmatrix},0,\ldots,0\right).$$

Note that the symmetric part of each 2-block is proportional to a 2-dimensional identity matrix, i.e. it is "shear-free".

Counterexample to the converse of Theorem:
5d type I_i Ricci flat example: direct product of 4d type I Ricci-flat cylindrical Newman-Tamburino solution

$$\mathrm{d}s^2 = r^2\mathrm{d}x^2 + x^2\mathrm{d}y^2 - \frac{4r}{x}\mathrm{d}u\mathrm{d}x - 2\mathrm{d}u\mathrm{d}r + x^{-2}\left(c + \ln(r^2x^4)\right)\mathrm{d}u^2,\ c = \text{const}$$

with a flat dimension

$l_a dx^a = \mathrm{d}u$
- geodesic PND of 4D spacetime,
optical matrix $\mathrm{diag}(b, 0)$,
- (**not multiple**) WAND of 5D spacetime,
optical matrix $\mathrm{diag}(b, 0, 0)$, i.e. case (iii) of Theorem with $a = 0$.

Therefore *the existence of a null geodesic congruence whose optical matrix takes the canonical form iii) of Theorem 1 is not a sufficient condition for the spacetime to be algebraically special.*

- Goldberg-Sachs theorem can be partially generalized to higher dimensions.

- Geodetic part: There exist algebraically special Einstein spacetimes with **non-geodetic** multiple WANDs. One can however show ([Durkee, Reall, CQG 2009]) that these spacetimes also always admit **geodetic** multiple WANDs.

- "Shearfree" part:
  - "**Generic**" type II and III and type N Weyl tensor $\Rightarrow$ **optical constraint** holds in arbitrary dimension $\Rightarrow$ $\boldsymbol{\rho}$ consists of **shearfree** 2×2 **blocks**.
  - To study all possible special cases one has to choose a specific low dimension. For $n = 5$ one arrives to **3 possible forms of** $\boldsymbol{\rho}$. Several examples for all these three classes are known. **While for** $n = 4$ **GS theorem reduces the number of free parameters from 3 to 2, in five dimensions the reduction is from 6 to 2**.