Evolution of the Einstein equations to future null infinity

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Relativity and Gravitation
100 Years after Einstein in Prague
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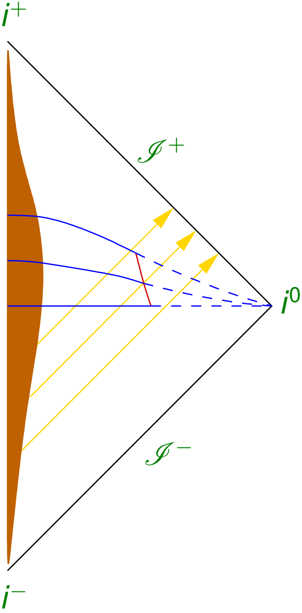
Cauchy evolution

- Standard approach to numerical evolutions of asymptotically flat spacetimes:
- Foliation by spacelike hypersurfaces approaching i^0, truncated at finite distance
- Need to impose boundary conditions
  - well posed
  - compatible with the constraints
  - absorbing
- Bad choice of boundary conditions can destroy relevant features of the solution
- Gravitational radiation only defined unambiguously at $\mathscr{I}^+$

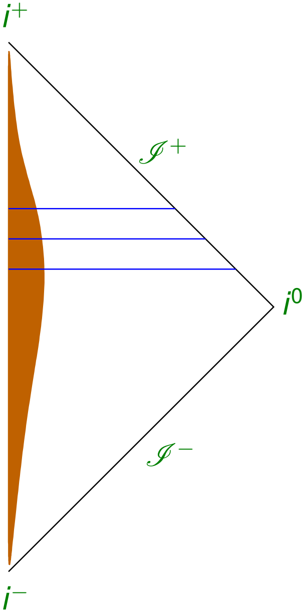


Hyperboloidal evolution

- Hyperboloidal hypersurfaces: spacelike but intersect $\mathscr{I}^+$
- Here: constant mean curvature (CMC)
- Conformal transformation of the metric

$$g_{ab} = \Omega^{-2}\tilde{g}_{ab}$$

  with $\Omega \searrow 0$ at $\mathscr{I}^+$
- Work with $\tilde{g}_{ab}$ in compactified coordinates
- Einstein equations contain formally singular terms at $\mathscr{I}^+$
- Alternative: regular conformal field equations (Friedrich 1983; numerical work: Hübner, Husa, Frauendiener, ...)



Conformal ADM decomposition

- Metric

$$
\begin{aligned}
g &= -N^2 dt^2 + \gamma_{ij}(dx^i + X^i dt)(dx^j + X^j dt) \\
&= \Omega^{-2}[-\tilde{N}^2 dt^2 + \tilde{\gamma}_{ij}(dx^i + X^i dt)(dx^j + X^j dt)]
\end{aligned}
$$

- Future-directed unit normal to $t =$ const slices: $n^a = \Omega \tilde{n}^a$
- Extrinsic curvature

$$K_{ij} = \tfrac{1}{2}\mathcal{L}_n \gamma_{ij}$$

- CMC slicing

$$\gamma^{ij} K_{ij} \equiv K = \text{const} > 0$$

- Traceless part

$$\pi^{\text{tr}\, ij} = \mu_\gamma(\gamma^{ik}\gamma^{jl} - \tfrac{1}{3}\gamma^{ij}\gamma^{kl})K_{kl},$$

where $\mu_\gamma = \sqrt{\det(\gamma_{ij})}$

Matter

- Define conformally rescaled energy-momentum tensor

$$\tilde{T}_{ab} = \Omega^{-2} T_{ab}$$

- Energy-momentum conservation equations transform as

$$\tilde{g}^{ab}\,\tilde{\nabla}_a \tilde{T}_{bc} = \Omega^{-4} g^{ab} \left(\nabla_a T_{bc} - \Omega^{-1} \nabla_c \Omega \, T_{ab} \right)$$

- Conformally invariant if we require the energy-momentum tensor to be tracefree, $g^{ab} T_{ab} = 0$
- Examples: conformally coupled scalar field, Maxwell, Yang-Mills, radiation fluid ($p = \frac{1}{3}\rho$), massless Einstein-Vlasov
- Define projections

$$\tilde{\rho} \equiv \tilde{n}^a \tilde{n}^b \tilde{T}_{ab}, \qquad \tilde{J}^i \equiv -\tilde{\gamma}^{ia} \tilde{n}^b \tilde{T}_{ab}, \qquad \tilde{S}_{ij} \equiv \tilde{\gamma}_i{}^a \tilde{\gamma}_j{}^b \tilde{T}_{ab}$$

Evolution equations

- ADM geometry evolution equations

$$\begin{aligned}
\mathcal{L}_{\tilde{n}}\tilde{\gamma}_{ij} &= 2\mu_{\tilde{\gamma}}^{-1}\tilde{\gamma}_{ik}\tilde{\gamma}_{jl}\pi^{\mathrm{tr}\,kl} + \tfrac{2}{3}\tilde{\gamma}_{ij}\tilde{K}, \\
\mathcal{L}_{\tilde{n}}\pi^{\mathrm{tr}\,ij} &= -2\mu_{\tilde{\gamma}}^{-1}\tilde{\gamma}_{kl}\pi^{\mathrm{tr}\,ik}\pi^{\mathrm{tr}\,jl} - \tfrac{2}{3}\Omega^{-1}K\pi^{\mathrm{tr}\,ij} \\
&\quad + \mu_{\tilde{\gamma}}\left[\tilde{N}^{-1}\tilde{D}^i\tilde{D}^j\tilde{N} - \tilde{R}^{ij} - 2\Omega^{-1}\tilde{D}^i\tilde{D}^j\Omega + \kappa\Omega^2\tilde{S}^{ij}\right]^{\mathrm{tr}},
\end{aligned}$$

  where $\tilde{D}$ is covariant derivative of $\tilde{\gamma}$, $\tilde{R}_{ij}$ Ricci tensor of $\tilde{\gamma}$, $\tilde{K}$ conformal mean curvature
- Matter evolution equations from $\tilde{\nabla}^a\tilde{T}_{ab} = 0$: regular at $\mathscr{I}^+$
- Goal: evaluate formally singular terms at $\mathscr{I}^+$ in evolution equation for $\pi^{\mathrm{tr}\,ij}$

Elliptic equations

- Hamiltonian constraint

$$-4\Omega\tilde{D}^i\tilde{D}_i\Omega+6\tilde{\gamma}^{ij}\Omega_{,i}\Omega_{,j}-\Omega^2\tilde{R}-\tfrac{2}{3}K^2+\Omega^2\mu_{\tilde{\gamma}}^2\tilde{\gamma}_{ik}\tilde{\gamma}_{jl}\pi^{\mathrm{tr}\,ij}\pi^{\mathrm{tr}\,kl}+2\kappa\Omega^4\tilde{\rho}=0$$

- Momentum constraints

$$\tilde{D}_j(\Omega^{-2}\pi^{\mathrm{tr}\,ij})+\kappa\mu_{\tilde{\gamma}}\tilde{J}^i=0$$

- CMC slicing condition

$$-\Omega^2\tilde{D}^i\tilde{D}_i\tilde{N}+3\Omega\tilde{\gamma}^{ij}\tilde{N}_{,i}\Omega_{,j}-\tfrac{3}{2}\tilde{N}\tilde{\gamma}^{ij}\Omega_{,i}\Omega_{,j}+\tfrac{1}{6}\tilde{N}K^2-\tfrac{1}{4}\tilde{N}\Omega^2\tilde{R}$$
$$+\tfrac{5}{4}\mu_{\tilde{\gamma}}^{-2}\tilde{\gamma}_{ik}\tilde{\gamma}_{jl}\pi^{\mathrm{tr}\,ij}\pi^{\mathrm{tr}\,kl}+\tfrac{1}{2}\kappa\tilde{N}\Omega^4(\tilde{S}+2\tilde{\rho})=0$$

- Spatial coordinate condition: e.g. harmonic $\Rightarrow$ elliptic eqn for X^i
- Conformal gauge condition: e.g. $\tilde{R}=$ const $\Rightarrow$ elliptic eqn for $\tilde{K}$

Regularity at future null infinity

- On a fixed spatial slice, choose coordinates $x^i = (x^1, x^A) = (r, \theta, \phi)$ with $r = r_+ =$ const at the cut with $\mathscr{I}^+$
- Expand the fields in *finite* Taylor series in r about r_+
- Substitute in singular elliptic equations and evaluate order by order
- Obtain expressions for up to third r-derivatives of Ω and up to first r-derivatives of $\pi^{\mathrm{tr}\, ri}$ at $\mathscr{I}^+$
- Recover necessary conditions for regularity at $\mathscr{I}^+$ (cf. Andersson, Chruściel & Friedrich 1992):

$$\pi^{\mathrm{tr}\, ri} \hat{=} 0,$$

$$\sigma^{AB} \equiv \mu_\gamma^{-1} \pi^{\mathrm{tr}\, AB} + \lambda^{\mathrm{tr}\, AB} \hat{=} 0 \qquad \text{(shear-free)}$$

  where λ_{AB} is second fundamental form of $r =$ const surfaces
- Regularity conditions preserved under time evolution
- Formally singular terms in evolution equation for $\pi^{\mathrm{tr}\, ij}$ can be evaluated explicitly at $\mathscr{I}^+$, *even if matter is included*

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Application: decay on Minkowski and black holes

- Successful hyperboloidal evolutions of test fields (Zenginoğlu *et al.*, Rácz & Tóth, Jasiulek, ...)
- Here: evolve using full nonlinear Einstein-matter equations

1. Vacuum, axisymmetry, perturbed Schwarzschild
   - Background solution: CMC slicing of Schwarzschild spacetime (Brill, Cavallo & Isenberg 1980)
   - Quasi-isotropic gauge

$$\tilde{\gamma} = e^{2\eta \sin\theta}(dr^2 + r^2 d\theta^2) + r^2 \sin^2\theta \, d\phi^2$$

   - Perturb η
2. Yang-Mills, spherical symmetry, decay to Minkowski
   - Isotropic gauge ($\tilde{\gamma}$ flat)
   - Yang-Mills gauge group SO(3), ansatz

$$\tilde{A}_i^{(a)} = \epsilon_{aij} x^j F(t, r), \quad A_0^{(a)} = 0$$

Numerics

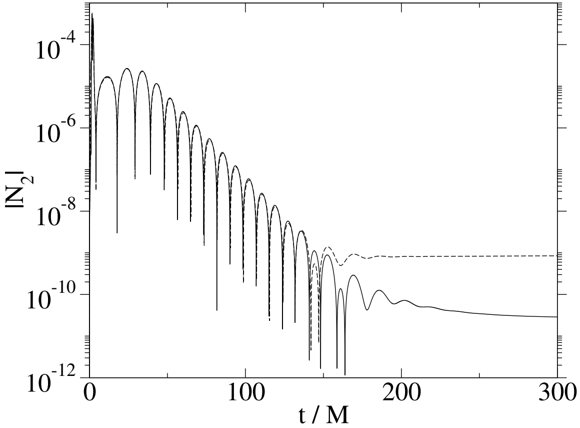
- Fourth-order finite differences
- Spherical polar coordinates, grid non-uniform in r
- Time integration: method of lines, fourth-order Runge-Kutta
- Fast elliptic solver
  - Spherical symmetry: direct band-diagonal solver + Newton iteration
  - Axisymmetry: nonlinear multigrid (FAS)

- Alternatively: pseudospectral method (under development)

Vacuum, axisymmetry, perturbed Schwarzschild

Initial perturbation with Gaussian shape, $A = 10^{-4}$, $r_0 = 0.5$, $\sigma = 0.05$

Bondi news function at $\mathscr{I}^+$



Quasi-normal modes

$$
\begin{aligned}
N_\ell &\propto e^{-\kappa_\ell t} \sin(\omega_\ell t + \varphi_\ell) \\
\kappa_2 &= 0.0893 \pm 0.0009 \\
(\kappa_2 &= 0.08896) \\
\omega_2 &= 0.3738 \pm 0.0012 \\
(\omega_2 &= 0.37367)
\end{aligned}
$$

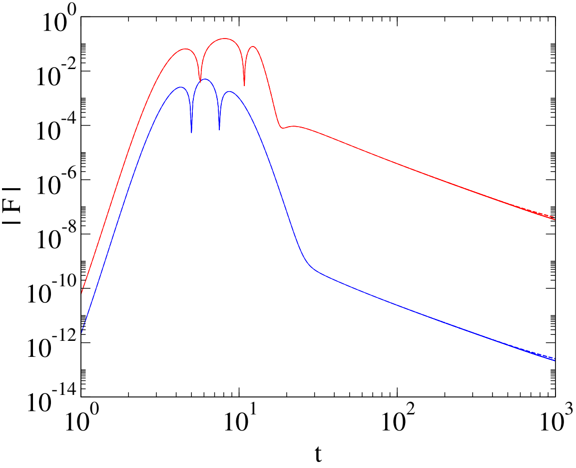
(results from linear perturbation theory in brackets)

Numerical resolution $(N_r, N_\theta) = (64, 8)$ (dashed) and $(128, 16)$ (solid)

Yang-Mills, spherical symmetry, decay to Minkowski

Initial perturbation with Gaussian shape, $r_0 = 0$, $\sigma = 0.1$

Yang-Mills potential F at $\mathscr{I}^+$



Blue: decoupled, $A = 1$

Red: coupled, $A = 29$
(initial Bondi mass ≈ 1)

Tail $F \sim t^{-2}$ in both cases

Numerical resolution $N_r = 500$ (dashed) and 1000 (solid)

Outlook

- Tails for decay to Schwarzschild
- Gravitational collapse
- Back to axisymmetry (and beyond?)