Turbulent Instability of Anti-de Sitter Space

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joint work with Piotr Bizoń
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Anti-de Sitter spacetime in $d+1$ dimensions

AdS is the maximally symmetric solution of the vacuum Einstein equations $R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R + \Lambda g_{\alpha\beta} = 0$ with negative cosmological constant $\Lambda < 0$:

$$ds^2 = -\left(1 + \frac{r^2}{\ell^2}\right)dt^2 + \frac{dr^2}{1 + r^2/\ell^2} + r^2 d\Omega^2_{S^{d-1}}, \qquad 0 \le r, \ -\infty < t < \infty$$

$$\Lambda = -d(d-1)/(2\ell^2)$$

Substituting $r = \ell \tan x$ we get

$$ds^2 = \frac{\ell^2}{(\cos x)^2}\left[-dt^2 + dx^2 + (\sin x)^2 d\Omega^2_{S^{d-1}}\right],$$

$$-\infty < t < +\infty, \quad 0 \le x < \pi/2.$$

Conformal infinity $x = \pi/2$ is the timelike hypersurface $\mathcal{I} = \mathbb{R} \times S^{d-1}$ with the boundary metric $ds^2_{\mathcal{I}} = -dt^2 + d\Omega^2_{S^{d-1}}$

Is AdS stable?

- A solution (of a dynamical system) is said to be stable if small perturbations of it at $t = 0$ remain small for all later times

- The question of stability of AdS is open. Surprisingly, with almost fifteen years of activity on AdS/CFT, this question has been rarely asked (with a notable exception M. Anderson 2005)

- In contrast, Minkowski ($\Lambda = 0$) and de Sitter spacetimes ($\Lambda > 0$) are known to be stable (actually asymptotically stable) – (Christodoulou&Klainerman 1993 and Friedrich 1986)

- The key difference between these solutions and AdS: the main mechanism of stability - dissipation of energy (dispersion in Minkowski, expansion in de Sitter) - is absent in AdS because AdS is effectively bounded (for no flux boundary conditions at $\mathcal{I}$ it acts as a perfect cavity)

- Note that by positive energy theorems both Minkowski and AdS are the unique ground states among asymptotically flat/AdS spacetimes

Model

- To deal with the problem of the stability of AdS we start with spherical symmetry (effectually $1+1$ dimensional problem)

- Spherically symmetric vacuum solutions are static (Birkhoff's theorem) $\Rightarrow$ we need matter to generate dynamics

- Simple matter model: massless scalar field ϕ in d+1 dimensions

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi G \left(\partial_\alpha \phi \, \partial_\beta \phi - \frac{1}{2} g_{\alpha\beta} \partial_\mu \phi \partial^\mu \phi \right), \, \Lambda = -d(d-1)/(2\ell^2)$$

$$g^{\alpha\beta} \nabla_\alpha \nabla_\beta \phi = 0$$

- In the corresponding asymptotically flat ($\Lambda = 0$) model Christodoulou proved the weak cosmic censorship (dispersion for small data and collapse to a black hole for large data) and Choptuik discovered critical phenomena at the threshold for black hole formation

- Remark: For even $d \geq 4$ there is a way to bypass Birkhoff's theorem (cohomogeneity-two Bianchi IX ansatz, Bizoń, Chmaj, Schmidt 2005)

- Convenient parametrization of asymptotically AdS spacetimes

$$ds^2 = \frac{\ell^2}{\left(\cos x\right)^2} \left[-A e^{-2\delta} dt^2 + A^{-1} dx^2 + \left(\sin x\right)^2 d\Omega^2_{S^{d-1}} \right] ,$$

where A and δ are functions of (t, x).

- Auxiliary variables $\Phi = \phi'$ and $\Pi = A^{-1} e^{\delta} \dot{\phi}$ $(' = \partial_x, \cdot = \partial_t)$
- Field equations (using units where $8\pi G = d - 1$)

$$A' = (1 - A) \frac{d - 2 + 2\left(\sin x\right)^2}{\left(\cos x\right)\left(\sin x\right)} - \left(\cos x\right)\left(\sin x\right) A \left(\Phi^2 + \Pi^2\right) ,$$

$$\delta' = - \left(\cos x\right)\left(\sin x\right) \left(\Phi^2 + \Pi^2\right) ,$$

$$\dot{\Phi} = \left(A e^{-\delta} \Pi\right)' , \qquad \dot{\Pi} = \frac{1}{\left(\tan x\right)^{d-1}} \left[\left(\tan x\right)^{d-1} A e^{-\delta} \Phi \right]' .$$

- AdS space: $\phi \equiv 0$, $A \equiv 1$, $\delta \equiv 0$; now we want to perturb AdS solving the initial-boundary value problem for this system starting with some small, smooth initial data $(\phi, \dot{\phi})_{|t=0}$

Boundary conditions

- We assume that initial data $(\phi, \dot{\phi})_{|t=0}$ are smooth
- Smoothness at the center implies that near $x = 0$

$$\phi(t, x) = f_0(t) + \mathcal{O}(x^2), \quad \delta(t, x) = \mathcal{O}(x^2), \quad A(t, x) = 1 + \mathcal{O}(x^2)$$

- Smoothness at spatial infinity and conservation of the total mass M imply that near $x = \pi/2$ (using $z = \pi/2 - x$)

$$\phi(t, x) = f_\infty(t)\, z^d + \mathcal{O}\left(z^{d+2}\right), \quad \delta(t, x) = \delta_\infty(t) + \mathcal{O}\left(z^{2d}\right),$$

$$A(t, x) = 1 - (M/\ell^{d-2}) z^d + \mathcal{O}\left(z^{d+2}\right)$$

Remark: There is no freedom in prescribing boundary data
- Local well-posedness (Friedrich 1995, Holzegel&Smulevici 2011)
- mass function and asymptotic mass:

$$m(t, x) = (1 - A(t, \rho)) \, (\ell \tan x)^{d-2} \, \left(1 + \tan^2 x\right)$$

$$M = \lim_{x \to \pi/2} m(t, x) = \ell^{d-2} \int_0^{\pi/2} \left(A\Phi^2 + A\Pi^2\right) (\tan x)^{d-1} \, dx$$
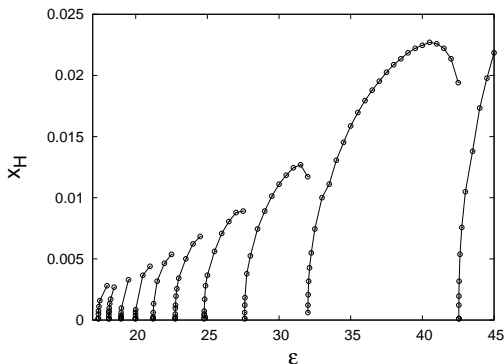
Movie

- from now $d = 3$
- qualitatively the same behaviour in any $d \geq 3$
- $d = 2$ case is very special (Pretorius&Choptuik 2000)

Critical behavior

Initial data: $\Phi(0, x) = 0$, $\Pi(0, x) = \varepsilon \left[\exp \left(-\frac{\tan x}{\sigma} \right)^2 \right]$

We fix $\sigma = 1/16$ and vary ε.

BH mass vs. amplitude

There is a decreasing sequence of critical amplitudes ε_n for which the evolution, after making n reflections from the AdS boundary, locally asymptotes Choptuik's solution. In each small right neighborhood of ε_n

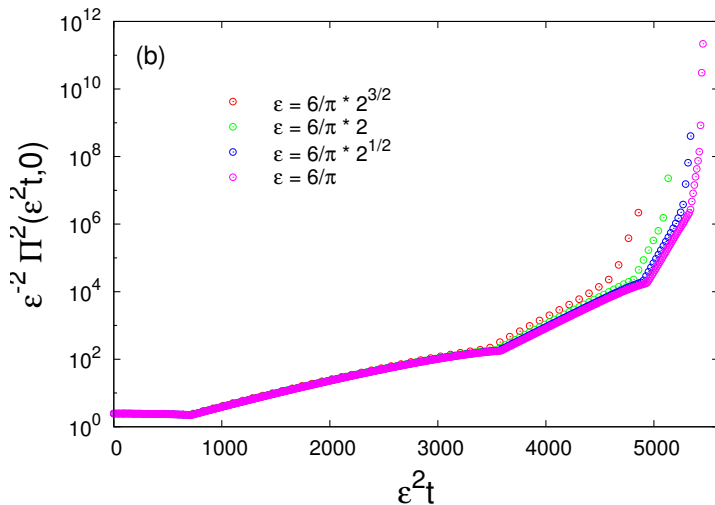$$m_{BH}(\varepsilon) \sim (\varepsilon - \varepsilon_n)^{\gamma}$$

with $\gamma \simeq 0.37$. It seems that $\lim_{n \to \infty} \varepsilon_n = 0$

Remark: The generic endstate of evolution is the Schwarzschild-AdS BH of mass M (in accord with Holzegel&Smulevici 2011)

Key evidence for instability
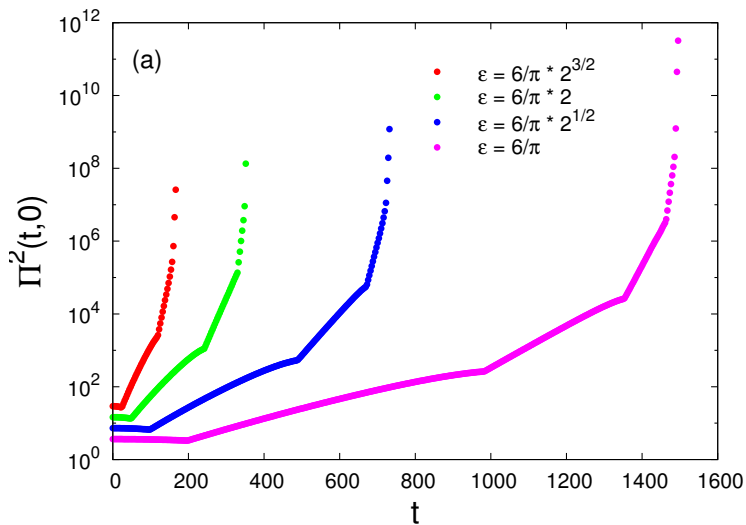
Ricci scalar $R = 2 \left(\Phi^2 - \Pi^2 \right) / \ell^2 - 12/\ell^2$

Key evidence for instability

Onset of instability at time $t = \mathcal{O}(\varepsilon^{-2})$

Weakly nonlinear perturbations

- We seek an approximate solution starting from small initial data
  $(\phi, \dot{\phi})_{|t=0} = (\varepsilon f(x), \varepsilon g(x))$
- Perturbation series

$$\phi = \varepsilon \phi_1 + \varepsilon^3 \phi_3 + ...$$
$$\delta = \varepsilon^2 \delta_2 + \varepsilon^4 \delta_4 + ...$$
$$1 - A = \varepsilon^2 A_2 + \varepsilon^4 A_4 + ...$$

where $(\phi_1, \dot{\phi}_1)_{|t=0} = (f(x), g(x))$ and $(\phi_j, \dot{\phi}_j)_{|t=0} = (0, 0)$ for $j > 1$.

- Inserting this expansion into the field equations and collecting terms of the same order in ε, we obtain a hierarchy of linear equations which can be solved order-by-order.

First order

- Linearized equation (Ishibashi&Wald 2004)

$$\ddot{\phi}_1 + L\phi_1 = 0, \quad L = -\frac{1}{\tan^2 x}\, \partial_x \left(\tan^2 x\, \partial_x\right)$$

The operator L is essentially self-adjoint on $L^2([0, \pi/2), \tan^2 x\, dx)$.

- Eigenvalues and eigenvectors of L are $(j = 0, 1, \dots)$

$$\omega_j^2 = (3 + 2j)^2, \quad e_j(x) = N_j (\cos x)^3\, F\left(\begin{array}{c} -j,\ 3+j \\ 3/2 \end{array} \middle|\, (\sin x)^2\right)$$

$\Rightarrow$ AdS is linearly stable

- Linearized solution

$$\phi_1(t, x) = \sum_{j=0}^{\infty} a_j \cos(\omega_j t + \beta_j)\, e_j(x)$$

where amplitudes a_j and phases β_j are determined by the initial data.

Second order (back-reaction on the metric)

$$A_2' + \frac{1 + 2\sin^2 x}{\sin x \cos x}\, A_2 = \sin x \cos x \left(\dot{\phi}_1^2 + {\phi_1'}^2\right)$$

$$\delta_2' = -\sin x \cos x \left(\dot{\phi}_1^2 + {\phi_1'}^2\right)$$

so

$$A_2(t,x) = \frac{\cos^3 x}{\sin x} \int\limits_0^x \left(\dot{\phi}_1(t,y)^2 + \phi_1'(t,y)^2\right) \tan^2 y \, dy$$

$$\delta_2(t,x) = -\int\limits_0^x \left(\dot{\phi}_1(t,y)^2 + \phi_1'(t,y)^2\right) \sin y \cos y \, dy$$

quadratic in ϕ_1!

$$(1)$$

It follows that
$$M = \frac{\varepsilon^2}{2} \int\limits_0^{\pi/2} \left(\dot{\phi}_1(t,y)^2 + \phi_1'(t,y)^2\right) \tan^2 y \, dy + \mathcal{O}(\varepsilon^4)$$

Third order

- $\ddot{\phi}_3 + L\phi_3 = S(\phi_1, A_2, \delta_2)$, $\qquad\qquad\qquad (\star)$
  where $S := 2(A_2 + \delta_2)\ddot{\phi}_1 + (\dot{A}_2 + \dot{\delta}_2)\dot{\phi}_1 + (A_2' + \delta_2')\phi_1'$.

- Let $\phi_3(t, x) = \sum_j c_j(t)\, e_j(x)$. Projecting Eq.($\star$) on the basis $\{e_j\}$ we obtain an infinite set of decoupled forced harmonic oscillators for the generalized Fourier coefficients $c_j(t) := (e_j, \phi_3)$

$$\ddot{c}_j + \omega_j^2 c_j = S_j := (e_j, S) \quad \text{and} \quad (c_j, \dot{c}_j)|_{t=0} = 0$$

  If S_j has a part oscilating with a resonant frequency ω_j it will give rise to a secular term in c_j, that is $c_j \sim t\sin\omega_j t$ or $c_j \sim t\cos\omega_j t$.

- S cubic in $\phi_1 \Rightarrow$ contains all frequencies $|\pm\omega_1 \pm\omega_2 \pm\omega_3|$, where $\omega_k \in \Omega_1$ and $\phi_1(t, x) = \sum_k [\omega_k \in \Omega_1]\, a_k\cos(\omega_k t + \beta_k)\, e_k(x)$,
  ω_k - odd integers $\Rightarrow$ all frequencies in S potentially resonant!
  Not all resonances survive the projection (e_j, S). Some of those, which do survive can be compensated with frequency shifts in ϕ_1 and are harmless for stability, but the others put stability in question!

Example 1: single-mode data $\phi(0, x) = \varepsilon\, e_0(x)$

- First order $\phi_1(t, x) = \cos(\omega_0 t) e_0(x)$, $\quad \omega_0 = 3 \qquad (\omega_j = 3 + 2j)$
- Third order $\phi_3(t, x) = \sum_{j=0}^{\infty} c_j(t) e_j(x)$, $\quad (c_j, \dot{c}_j)|_{t=0} = 0$ and

$$\ddot{c}_j + \omega_j^2 c_j = b_{j,0} \cos(\omega_0 t) + b_{j,3} \cos(\omega_3 t).$$

But $b_{3,3} = 0$ (!) and only $j = 0$ is resonant.
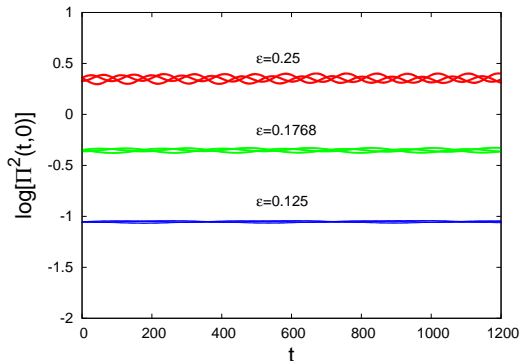
The $j = 0$ resonance can be easily removed by the two-scale method (slow-time phase modulation) which gives $\phi_1 = \cos((\omega_0 + \frac{153}{4\pi}\varepsilon^2)\, t)\, e_0(x)$. This suggests that there are non-generic initial data which may stay close to AdS solution

Example 2: two-mode data $\phi(0, x) = \varepsilon \left(e_0(x) + e_1(x) \right)$

- First order $\phi_1(t, x) = \cos(\omega_0 t) e_0(x) + \cos(\omega_1 t) e_1(x)$, $\omega_0 = 3$, $\omega_1 = 5$
- Third order $\phi_3(t, x) = \sum_{j=0}^{\infty} c_j(t) e_j(x)$, $(c_j, \dot{c}_j)|_{t=0} = 0$ and

$$\ddot{c}_j + \omega_j^2 c_j = \sum_k [\omega_k \in \Omega_3] b_{j,k} \cos(\omega_k t), \quad \text{where } \Omega_3 = \{|\pm\omega_{0,1} \pm \omega_{0,1} \pm \omega_{0,1}$$

Here $\Omega_3 = \{1, 3, 5, 7, 9, 11, 13, 15\}$, but the resonance ($b_{j,j} \neq 0$) only if $\omega_j \in \{3, 5, 7\}$.
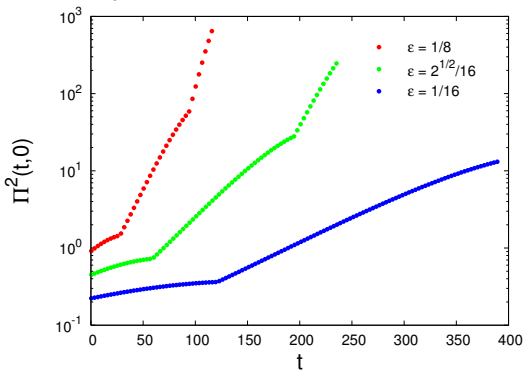
$\omega_0 \to \omega_0 + (87/\pi)\epsilon^2$, $\omega_1 \to \omega_1 + (413/\pi)\epsilon^2$ shifts remove the resonances $\omega_j = 3, 5$, but the resonance $\omega_j = 7$ cannot be removed. Thus we get the secular term $c_2(t) \sim t \sin(7t)$. We expect this term to be a progenitor of the onset of exponential instability.

Conjectures

Our numerical and formal perturbative computations lead us to:

Conjecture 1

Anti-de Sitter space is unstable against the formation of a black hole under arbitrarily small generic perturbations

Proof (and a precise formulation) is left as a challenge. Note that we do not claim that all perturbed solutions end up as black holes.

Conjecture 2

There are non-generic initial data which may stay close to AdS solution; Einstein-scalar-AdS equations may admit time-quasiperiodic solutions

Proof: KAM theory for PDEs?
Analogous conjecture for vacuum Einstein's equations by Dias, Horowitz and Santos (2011) (existence of geons)

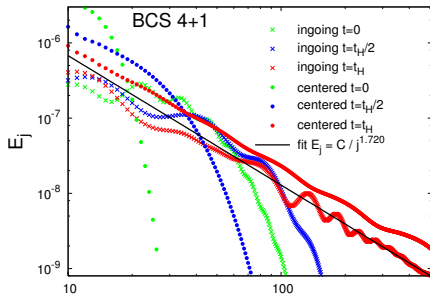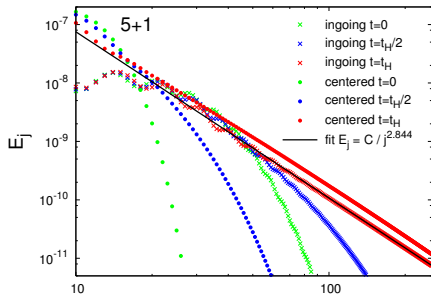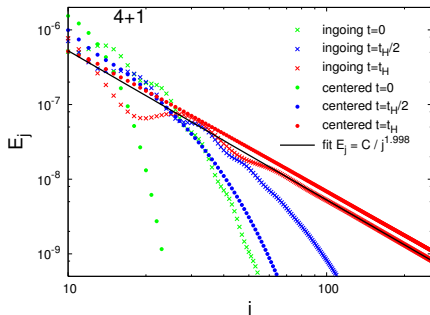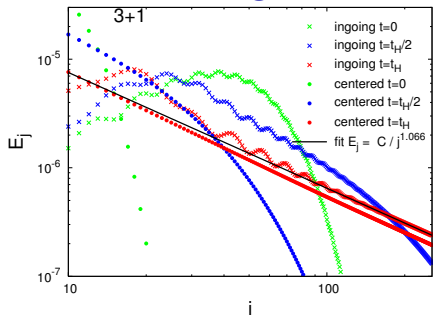# Turbulence: transfer of energy from low to high frequencies

Let $\Pi_j := (\sqrt{A}\,\Pi, e_j)$ and $\Phi_j := (\sqrt{A}\,\Phi, e_j')$. Then

$$M = \ell \int_0^{\pi/2} \left(A\Phi^2 + A\Pi^2\right)(\tan x)^2 \, dx = \sum_{j=0}^{\infty} E_j(t),$$

where $E_j := \Pi_j^2 + \omega_j^{-2}\Phi_j^2$ can be interpreted as the j-mode energy.
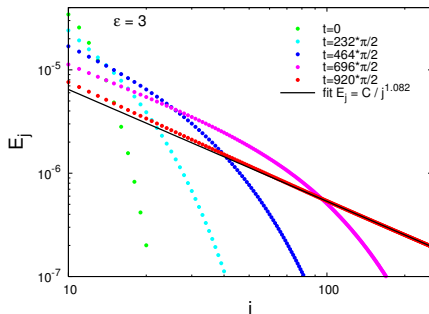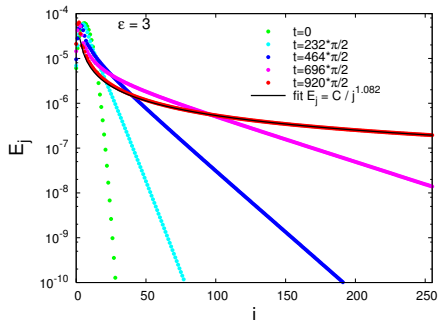
Power-law scaling

Final remarks

- Weakly turbulent behavior seems to be common for (non-integrable) nonlinear wave equations on bounded domains (e.g. NLS on torus, Colliander,Keel, Staffilani,Takaoka,Tao 2008, Carles,Faou 2010) and our work shows that Einstein's equations are not an exception.

- For Einstein's equations the transfer of energy to high frequencies cannot proceed forever because concentration of energy on smaller and smaller scales inevitably leads to the formation of a black hole.

- We believe that the role of negative cosmological constant is purely kinematical, that is the only role of Λ is to confine the evolution in an effectively bounded domain. Similar turbulent dynamics has been observed for small perturbations of Minkowski in a box (Maliborski)

- Generalizations: different matter models, "deformed" boundary conditions, relaxing symmetry, multi-scale analysis, (in)stability of AdS black holes (Holzegel&Smulevici 2011, Horowitz 2012), instability of AdS_{2+1} (work in progress by Jałmużna)