On the stability operator for MOTS and the 'core' of Black Holes

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Outline

Basic concepts and notation

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- I will also use the concept of OTS ($\theta_k < 0$) and of TS ($\theta_k < 0$ and $\theta_\ell < 0$).

The stability operator for MOTS

- As proven in (Andersson-Mars-Simon , Adv. Theor. Math. Phys. 12 (2008) 853), the variation of the vanishing expansion $\delta_{f\vec{n}}\theta_{\vec{k}}$ along any normal direction $f\vec{n}$ such that $k_\mu n^\mu = 1$ reads

$$\delta_{f\vec{n}}\theta_{\vec{k}} = -\Delta_S f + 2s^B \overline{\nabla}_B f + f\left(K_S - s^B s_B + \overline{\nabla}_B s^B - G_{\mu\nu}k^\mu \ell^\nu|_S - \frac{n^\rho n_\rho}{2} W\right) \quad (1)$$

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- Here K_S is the Gaussian curvature on S, Δ_S its Laplacian, $G_{\mu\nu}$ the Einstein tensor, $\overline{\nabla}$ the covariant derivative on S, $s_B = k_\mu e^\sigma_B \nabla_\sigma \ell^\rho$ (with $\vec{e}_B$ the tangent vector fields on S), and

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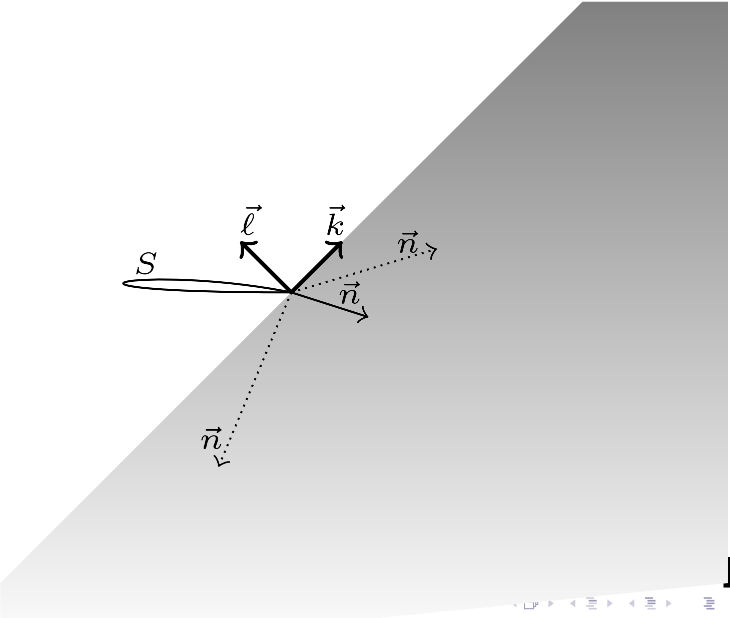
- Note that the direction $\vec{n}$ is selected by fixing its norm:

$$\vec{n} = -\vec{\ell} + \frac{n_\mu n^\mu}{2}\vec{k} \tag{3}$$

Observe also that the causal character of $\vec{n}$ is unrestricted.

Scheme for the variation direction



- Notice that $W \geq 0$ if $G_{\mu\nu}k^\mu k^\nu|_S \geq 0$ (for instance if NCC holds). Under this hypothesis, $W = 0$ can only happen if $G_{\mu\nu}k^\mu k^\nu|_S = \sigma^2 = 0$. This leads to Isolated Horizons, and I shall assume $W > 0$ throughout.

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- $L_{\vec{n}}$ is not self-adjoint in general. Nevertheless, it has a real principal eigenvalue $\lambda_{\vec{n}}$, and the corresponding (real) eigenfunction can be chosen to be positive on S.
- The (strict) stability of the MOTS S is ruled by the (positivity) non-negativity of $\lambda_{\vec{n}}$.

Spherically symmetric spacetimes as a Lab for $L_{\vec{n}}$

In advanced coordinates

$$ds^2 = -e^{2\alpha}\left(1 - \frac{2m(v,r)}{r}\right)dv^2 + 2e^{\alpha}dvdr + r^2 d\Omega^2$$

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- The null expansions:

$$\theta^{sph}_{\vec{k}} = \frac{e^{\alpha}}{r}\left(1 - \frac{2m}{r}\right), \qquad \theta^{sph}_{\vec{\ell}} = -\frac{2e^{-\alpha}}{r}.$$

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- However, how much must a TS penetrate into $\{r < 2m\}$?

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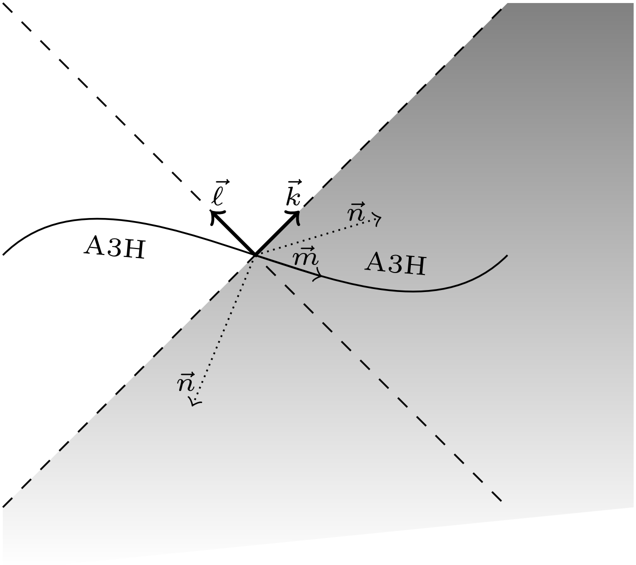
- Let us call such a vector field $\vec{m}$, so that $\vec{m} = -\vec{\ell} + \frac{m_\mu m^\mu}{2}\vec{k}$ with

$$\frac{1}{r_\varsigma^2} - G_{\mu\nu}k^\mu\ell^\nu|_\varsigma - \frac{m_\rho m^\rho}{2}G_{\mu\nu}k^\mu k^\nu\Big|_\varsigma = 0$$

characterizes A3H.

A helpful picture



Deformations on A3H\A3H^{iso}

- Consider now the parts of A3H with $G_{\mu\nu}k^\mu k^\nu > 0$. From the **helpful figure** we deduce that the perturbation along $f\vec{n}$ will enter into the region with trapped round spheres at points with

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- For easy control of these signs we note that

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- For this choice the deformed surface enters the region $\{r < 2m\}$ at points with $f > 0$.

TSs entering $\{r > 2m\}$

- Now set $f \equiv a_0 + \tilde{f}$ for some as yet undetermined function $\tilde{f}$ and a constant a_0. Equation (4) becomes

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- The second is a (mild) restriction on the function $\tilde{f}$. A simple solution is to choose $\tilde{f}$ to be an eigenfunction of the Laplacian Δ_ς, say
$$\tilde{f} = c_l P_l$$
for a fixed $l \in \mathbb{N}$ and constant c_l (P_l = Legendre polynomials).

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- If we choose a sufficiently small constant a_0 the last requirement implies that the region where the surface extends outside $\{r > 2m\}$ can be made arbitrarily small.
- To find such a function it is convenient to introduce stereographic coordinates $\{\rho, \varphi\}$ on the sphere, so that the Laplacian takes the form

$$\Delta_\varsigma = \Omega^{-1}\left(\partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \frac{1}{\rho^2}\partial_\varphi^2\right) \; , \qquad \Omega = \frac{4r_\varsigma^2}{(1+\rho^2)^2}.$$

An explicit solution to the problem

- A solution to the problem as stated is the axially symmetric function

$$\tilde{f}(\rho) = \begin{cases} c_1 \left(e^{\frac{1}{2a}(2a-\rho^2)} - 1 \right) & \rho^2 < 4a \\ \frac{8c_1 a}{e} \frac{1}{\rho^2} - c_1(1 + e^{-1}) & \rho^2 > 4a \ . \end{cases} \tag{5}$$

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- This function is C^2 (and can be further smoothed if necessary), and it is positive only if $\rho^2 < 2a$, that is on a disk surrounding the origin (the pole) whose size can be chosen at will.

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$$\tilde{f}(\rho) = \begin{cases} c_1 \left(e^{\frac{1}{2a}(2a-\rho^2)} - 1 \right) & \rho^2 < 4a \\ \frac{8c_1 a}{e} \frac{1}{\rho^2} - c_1(1+e^{-1}) & \rho^2 > 4a \ . \end{cases} \tag{5}$$

- This function is C^2 (and can be further smoothed if necessary), and it is positive only if $\rho^2 < 2a$, that is on a disk surrounding the origin (the pole) whose size can be chosen at will.
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- This is always larger than zero.

A surprising theorem

Thus we have proven the following important result.

Theorem (Bengtsson & JMMS 2011)

In spherically symmetric spacetimes, there are closed f-trapped surfaces (topological spheres) penetrating both sides of the apparent 3-horizon A3H$\setminus$A3H^{iso} with arbitrarily small portions outside the region $\{r > 2m\}$.

The future-trapped region $\mathscr{T}$ and its boundary $\mathscr{B}$

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One of the mysteries concerning closed TSs is: where is $\mathscr{B}$? But this is another story....

Non-locality and clairvoyance of TSs

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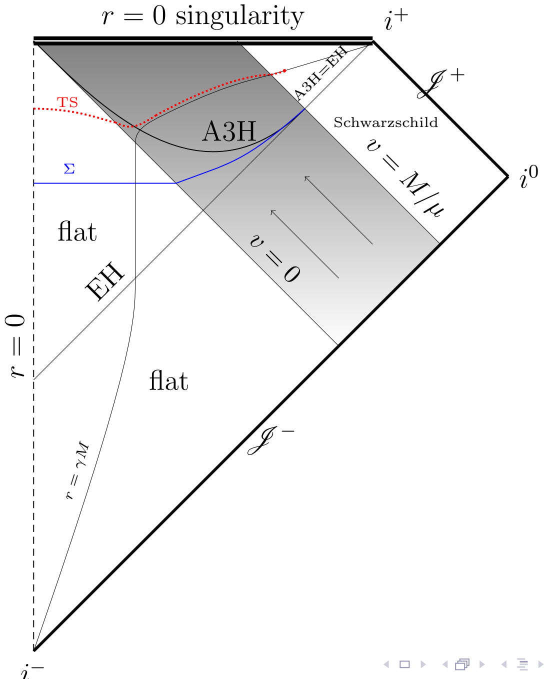
- What is?


r = 0 singularity
i+
A3H=EH
J+
TS
Schwarzschild
A3H
v = M/μ
Σ
i0
flat
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The Core of the trapped region

Definition

A region $\mathscr{Z}$ is called the core of the f-trapped region $\mathscr{T}$ if it is a minimal closed connected set that needs to be removed from the spacetime in order to get rid of all closed f-trapped surfaces in $\mathscr{T}$, and such that any point on the boundary $\partial\mathscr{Z}$ is connected to $\mathscr{B} = \partial\mathscr{T}$ in the closure of the remainder.

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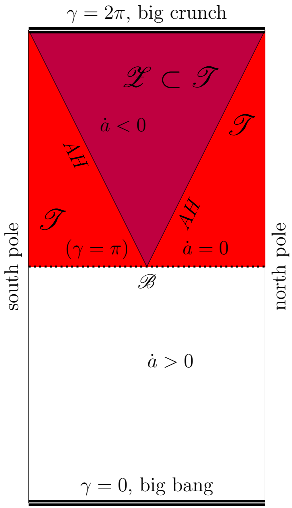
- Here, "minimal" means that there is no other set $\mathscr{Z}'$ with the same properties and properly contained in $\mathscr{Z}$.
- The final technical condition states that the excised space-time $(\mathcal{V}\backslash\mathscr{Z}, g)$ has the property that $\forall x \in \mathcal{V}\backslash\mathscr{Z} \cup \partial\mathscr{Z}$ there is continuous curve $\gamma \subset \mathcal{V}\backslash\mathscr{Z} \cup \partial\mathscr{Z}$ joining x and $\mathscr{B}$ (γ can have zero length if $\mathscr{B} \cap \partial\mathscr{Z} \neq \emptyset$).
  This is needed because one could identify a particular removable region, excise it, but then put back a tiny isolated portion to make it smaller. However, this is not what one wants to cover with the definition.

$\mathscr{X} \subset \mathscr{T}$. Example: RW, $p = 0$ and $\Lambda = 0$.



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- Actually this kind of non-uniqueness is rather trivial, and is due to the existence of a high degree of symmetry.
- Nevertheless, even in less symmetric cases the uniqueness of the cores $\mathscr{Z}$ cannot be assumed beforehand. Actually, we have proven that it does not hold in general (see below).

Cores in spherical symmetry

Result (Bengtsson & JMMS , 2011)

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Result

In spherically symmetric spacetimes, $\mathscr{Z} = \{r \leq 2m\}$ *are the only spherically symmetric cores of* $\mathscr{T}$. *Therefore,* $\partial\mathscr{Z} =$ *A3H are the only spherically symmetric boundaries of a core.*

Non-spherically symmetric cores

Proposition

There exist non-spherically symmetric cores of the f-trapped region in spherically symmetric spacetimes.

- Still, the identified core $\mathscr{Z} = \{r \leq 2m\}$ might be unique in the sense that its boundary $\partial\mathscr{Z} =$ A3H is a MTT.

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- Then A3H would be selected as the unique MTT which is the boundary of a core of the f-trapped region $\mathscr{T}$.
- Whether or not this happens is a very interesting open question.
- It should be observed that the concept of core is global, and requires full knowledge of the future. However, AH is local and can be defined and identified by observing just around it. How can then A3H $= \partial\mathscr{Z}$?

Back to the general case

- Recall the stability operator:

$$L_{\vec{n}} f = -\Delta_S f + 2s^B \overline{\nabla}_B f + f\left(K_S - s^B s_B + \overline{\nabla}_B s^B - G_{\mu\nu} k^\mu \ell^\nu|_S - \frac{n^\rho n_\rho}{2} W\right)$$

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- L_z has a principal *real* eigenvalue λ_z —which depends on z— and the corresponding eigenfunction $\phi_z > 0$.
- The variation of $\theta_{\vec{k}} = 0$ along the direction $\phi_z \vec{n}$ becomes

$$\frac{L_{\vec{n}} \phi_z}{\phi_z} = \lambda_z - z + K_S - s^B s_B + \overline{\nabla}_B s^B - G_{\mu\nu} k^\mu \ell^\nu |_S - \frac{n^\rho n_\rho}{2} W$$

Many MOTTs through a given MOTS

- Thus, *whenever* $W \neq 0$ on S, one can choose for any z a variation vector $\vec{m}_z = -\vec{\ell} + M_z \vec{k}$ with

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$$\delta_{f\vec{m}_z} \theta_{\vec{k}} = -\Delta_S f + 2s^B \overline{\nabla}_B f + f(z - \lambda_z) = (L_z - \lambda_z) f \quad (7)$$

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  so that the stability operator $L_{\vec{m}_z}$ of S along $\vec{m}_z$ is simply $L_z - \lambda_z$ which obviously has a vanishing principal eigenvalue.
- The directions $\vec{m}_z$ define locally MOTTs including any given *stable* MOTS S —due to a result in (Andersson-Mars-Simon 2005).

Many different MOTTs

- These MOTTs will generically be different for different z. In fact, given that $\forall z_1, z_2 \in C^\infty(S)$

$$\vec{m}_{z_1} - \vec{m}_{z_2} = \frac{1}{W}\left(\lambda_{z_1} - z_1 - \lambda_{z_2} + z_2\right)\vec{k}$$

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- Now, for any given z rewrite $\delta_{f\vec{n}}\theta_{\vec{k}} = L_{\vec{n}} f$ using (6) so that

$$\frac{W}{2} f\left(n^\rho n_\rho - m_z^\rho m_{z\rho}\right) = (L_z - \lambda_z) f - \delta_{f\vec{n}}\theta_{\vec{k}} \qquad (8)$$

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  3. and that $\lambda_z - \left(z - 2\overline{\nabla}_B s^B\right)$ must vanish somewhere on S for all $z \in C^\infty(S)$.

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- For such a choice let
  1. $L = L_{2\overline{\nabla}_B s^B}$,
  2. μ its principal eigenvalue,
  3. and $\phi > 0$ the corresponding eigenfunction.

- Observe that

$$Lf = -\Delta_S f + 2\overline{\nabla}_B(f s^B) = -\overline{\nabla}_B\left(\overline{\nabla}^B f - 2 f s^B\right).$$

- The principal eigenvalue μ vanishes. Indeed, this follows immediately from either (9) or (10). Also from

$$\oint_S Lf = 0 \quad \forall f, \Longrightarrow \oint_S L\phi = \mu \oint_S \phi = 0$$

A distinguished MOTT

- For this particular choice $z = 2\overline{\nabla}_B s^B$, (8) reduces to

$$\frac{W}{2} f \left(n^\rho n_\rho - m^\rho m_\rho\right) = Lf - \delta_{f\vec{n}} \theta_{\vec{k}} \tag{11}$$

where now the vector $\vec{m} = -\vec{\ell} + \frac{m^\rho m_\rho}{2} \vec{k}$ is defined by

$$\frac{m^\rho m_\rho}{2} = \frac{1}{W} \left(K_S - \overline{\nabla}_B s^B - s^B s_B - G_{\mu\nu} k^\mu \ell^\nu |_S \right)$$

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- For any other direction $\vec{m}_z$ defining a local M(O)TT

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Result

The local M(O)TT defined by the direction $\vec{m}$ is such that any other nearby local M(O)TT must interweave it with non-trivial intersections to both of its sides, that is to say, the vector $\vec{m}_z - \vec{m}$ changes sign on any of its M(O)TSs.

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for a constant $a_0 > 0$ so that, as $\phi > 0$ has eigenvalue $\mu = 0$, (11) becomes

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- Eq.(12) tells us that $\delta_{f\vec{n}}\theta_{\vec{k}} < 0$ whenever $\vec{n}$ points "above" $\vec{m}$ if $a_0 > 0$ is chosen.

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- To prove that there are future-trapped surfaces penetrating both sides of the MTT it is enough to comply with points 1 and 2 only.
- This would certainly happen if L has more real eigenvalues, and leads to the analysis of the condition $L\tilde{f}/\tilde{f} > 0$ for some function $\tilde{f}$.

The case when L has real eigenvalues

Result

If the operator L has any real eigenvalue other than the principal one $\mu = 0$, then the conditions 1 and 2 do hold for the corresponding real eigenfunction. This leads to the existence of closed OTSs penetrating both sides of the local M(O)TT.

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Proof. Any real eigenvalue is strictly positive (as $\mu = 0$). Hence, the corresponding eigenfunction must change sign on S, because integration of $L\psi = \lambda\psi$ on S implies $\oint \psi = 0$.

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However, even if there are no other real eigenvalues the result might hold.