The twin paradox in static spacetimes and Jacobi fields

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The twin paradox in SR has 2 levels of understanding:
— why the effect is asymmetric at all (the relativity principle suggests a symmetry of the twins),
— why the twin in non-uniform motion is always younger at the reunion.
Most textbooks: only the simplest case — one twin at rest and the other moving uniformly back and forth to a distant star $\Rightarrow$ explanation on the first, lowest level of understanding.

The second level of understanding requires:
physical time = proper time = length of the twin's worldline.
The twins A and B travel at arbitrary velocities in some inertial frame from point P to Q in Minkowski spacetime

$$s_A = \int_P^Q ds_A = c \int_{t_1}^{t_2} \sqrt{1 - \left(\frac{\mathbf{v}_A}{c}\right)^2}\, dt,$$

$$s_B = \int_P^Q ds_B = c \int_{t_1}^{t_2} \sqrt{1 - \left(\frac{\mathbf{v}_B}{c}\right)^2}\, dt \neq s_A.$$

In Minkowski spacetime:
the timelike straight line ($\mathbf{v} = \mathbf{0}$) from P to Q is the longest timelike curve, a timelike curved line may be arbitrarily short, $s(P, Q)$ close to 0.

Twin paradox in curved spacetimes

Special cases studied in some spacetimes (Schwarzschild, de Sitter).
Problem in static spacetimes: what makes one twin younger than the other — velocity (with respect to a static observer) or acceleration?
Actually: a multitude of possibilities and results found in first works are not generally valid.
The problem is purely geometrical.
Feynman at Princeton ('1940) was among the first who gave a correct, imprecise answer: the longest worldline is a timelike geodesic.
General rigorous answer:
S. Hawking and G. Ellis, *The large scale structure of space–time*, 1973.

PROPOSITION 1

In any convex normal neighbourhood U, if p and q can be joined by a timelike curve then the unique timelike geodesic connecting them has length strictly greater than that of any other piecewise smooth timelike curve between the points.

The existence of a convex normal neighbourhood is crucial here. If q does not lie in a normal neighbourhood of p then there are several timelike geodesics from p to q with different lengths.

Example: Schwarzschild BH

The twin A rotates on a circular geodesic orbit around the BH at $r = r_0$. The length of geodesic A after making one full circle is (P_0 – the starting point, P_1 – the endpoint in the spacetime)

$$s_A = \int_{P_0}^{P_1} ds_A = 2\pi r_0 \left(\frac{r_0 - 3M}{M} \right)^{1/2},$$

The twin B moves outwards on a radial geodesic emanating from P_0 with initial velocity $dr/ds = u > 0$ at $r = r_0$, reaches a maximal height $r = r_M$ and radially falls down to r_0 at P_1.
The radial geodesic B has length $s_B > s_A$.
Conclusion:
the spacetime region outside the event horizon is NOT a convex normal neighbourhood of its points.

Conjugate points

How to recognize whether given p and q can be connected by a *unique* timelike geodesic?

Key notion: *conjugate points*.

Let $Z^\mu(s)$ – a geodesic deviation vector field on a timelike geodesic γ with tangent unit u^α and $Z^\mu u_\mu = 0$, satisfies

$$\frac{D^2}{ds^2} Z^\mu = R^\mu{}_{\alpha\beta\gamma}\, u^\alpha\, u^\beta\, Z^\gamma .$$

Any solution Z^μ – Jacobi field on γ.

Definition:

points p and q on γ are *conjugate* if there is Jacobi field $Z^\mu(s) \neq 0$ such that $Z^\mu(q) = 0$ iff $Z^\mu(p) = 0$.

p and q are conjugate $\Rightarrow$ infinitesimally close geodesics intersect γ at p and q $\Leftrightarrow$ there are many timelike geodesics joining p and q.

PROPOSITION 2

A timelike geodesic has the maximal length from p to q iff there is no point conjugate to p on the segment pq.

Existence of conjugate points.

PROPOSITION 3

If $R_{\alpha\beta}\, u^\alpha\, u^\beta \geq 0$ on a timelike geodesic γ and if the tidal force $R_{\mu\alpha\nu\beta}\, u^\alpha\, u^\beta \neq 0$ at some point p_0 on γ, there will be conjugate points p and q somewhere on γ (providing that the geodesic can be extended sufficiently far).

Conclusion:

The problem of which twin will be older at the reunion has a generic solution only if one of the twins' worldlines is a timelike geodesic free of conjugate points.

In all other situations one must study case by case: compute lengths of various worldlines in a given spacetime.

Investigations

Behaviour of bunches of timelike geodesics (diverge or converge) is one of the most important geometric features of any spacetime.
Motivated by the twin paradox we study the Jacobi fields in various spacetimes.
Assumptions.
1. Geodesic lines are explicitly known $\Rightarrow$ we consider physically preferred curves (radial, circular).
2. Geodesic deviation equation is very complicated$\Rightarrow$ may be solved only if its first integrals exist. These exist if the spacetime has symmetries (Killing fields).

First results

1. In all the static spacetimes under consideration the radial geodesics are the longest and except de Sitter space contain conjugate points outside the relevant segment. There are either two or an infinite sequence of conjugate points.
2. In all static spherically symmetric spacetimes the circular geodesics (if exist) have the same property: there exist two infinite sequences (the same in all spacetimes) of conjugate points if the geodesic extends to infinity in the future and past.