$f(R)$ gravity - the most straightforward generalization of the Einstein gravity

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Why modified gravity?

$f(R)$ gravity

Inflationary models in $f(R)$ gravity

Present DE models in $f(R)$ gravity

Complete models of present DE in $f(R)$ gravity

Combined models of primordial and present DE

Conclusions

Why modified gravity?

Why going beyond the pure Einstein gravity (GR) interacting with dust (baryons + CDM) and radiation? – Existence of dark energy (DE).

Two cases where DE shows itself:
1) inflation in the early Universe – primordial DE,
2) present accelerated expansion of the Universe – present DE.

The whole known part of the history of our Universe in one line, according to the standard cosmological scenario:

$$? \longrightarrow DS \Longrightarrow FLRWRD \Longrightarrow FLRWMD \Longrightarrow \overline{DS} \longrightarrow ?$$

Remarkable qualitative similarity of DS and $\overline{DS}$ makes possible (though not necessary) combined description of both DS stages (both types of DE) using one class of models.

Possible forms of DE

- Physical DE
  New non-gravitational field of matter. DE proper place – in the rhs of gravity equations.
- Geometrical DE
  Modified gravity. DE proper place – in the lhs of gravity equations.
- Λ - intermediate case.

Generically, DE can be both physical and geometrical, e.g. in the case of a non-minimally coupled scalar field or, more generically, in scalar-tensor gravity. So, there is no alternative "(either) dark energy or modified gravity".

$f(R)$ gravity

The simplest model of modified gravity (= geometrical dark energy) considered as a phenomenological macroscopic theory in the fully non-linear regime and non-perturbative regime.

$$S = \frac{1}{16\pi G}\int f(R)\sqrt{-g}\, d^4x + S_m$$

$$f(R) = R + F(R), \quad R \equiv R_{\mu}^{\mu}$$

Here $f''(R)$ is not identically zero. Usual matter described by the action S_m is minimally coupled to gravity.

Vacuum one-loop corrections depending on R only (not on its derivatives) are assumed to be included into $f(R)$. The normalization point: at laboratory values of R where the scalaron mass (see below) $m_s \approx \mathrm{const}$.

Metric variation is assumed everywhere. Palatini variation leads to a different theory with a different number of degrees of freedom.

Field equations

$$\frac{1}{8\pi G}\left(R_\mu^\nu - \frac{1}{2}\,\delta_\mu^\nu R\right) = -\left(T_{\mu\,(vis)}^{\nu} + T_{\mu\,(DM)}^{\nu} + T_{\mu\,(DE)}^{\nu}\right)\ ,$$

where $G = G_0 = const$ is the Newton gravitational constant measured in laboratory and the effective energy-momentum tensor of DE is

$$8\pi G T_{\mu\,(DE)}^{\nu} = F'(R)\,R_\mu^\nu - \frac{1}{2}\,F(R)\delta_\mu^\nu + \left(\nabla_\mu\nabla^\nu - \delta_\mu^\nu\nabla_\gamma\nabla^\gamma\right)F'(R)\,.$$

Because of the need to describe DE, de Sitter solutions in the absence of matter are of special interest. They are given by the roots $R = R_{dS}$ of the algebraic equation

$$Rf'(R) = 2f(R)\ .$$

Degrees of freedom

I. In quantum language: particle content.
1. Graviton – spin 2, massless, transverse traceless.
2. Scalaron – spin 0, massive, mass - R-dependent:
$m_s^2(R) = \frac{1}{3f''(R)}$ in the WKB-regime.

II. Equivalently, in classical language: number of free functions of spatial coordinates at an initial Cauchy hypersurface.
Six, instead of four for GR – two additional functions describe massive scalar waves.

Thus, $f(R)$ gravity is a non-perturbative generalization of GR.
It is equivalent to scalar-tensor gravity with $\omega_{BD} = 0$ if $f''(R) \neq 0$.

Transformation to the Einstein frame and back

In the Einstein frame, free particles of usual matter do not follow geodesics and atomic clocks do not measure proper time.

From the Jordan (physical) frame to the Einstein one:

$$g^E_{\mu\nu} = f' g^J_{\mu\nu}, \quad \kappa\phi = \sqrt{\frac{3}{2}} \ln f', \quad V(\phi) = \frac{f'R - f}{2\kappa^2 f'^2}$$

where $\kappa^2 = 8\pi G$.

Inverse transformation:

$$R = \left(\sqrt{6}\kappa \frac{dV(\phi)}{d\phi} + 4\kappa^2 V(\phi)\right) \exp\left(\sqrt{\frac{2}{3}}\kappa\phi\right)$$

$$f(R) = \left(\sqrt{6}\kappa \frac{dV(\phi)}{d\phi} + 2\kappa^2 V(\phi)\right) \exp\left(2\sqrt{\frac{2}{3}}\kappa\phi\right)$$

$V(\phi)$ should be at least C^1.

Why R-dependence only?

For almost all other geometric invariants – $R_{\mu\nu}R^{\mu\nu}$, $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$, $R_{,\mu}R^{,\mu}$ etc. (where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor) – ghosts appear if the theory is taken in full, in the non-perturbative regime.

The only known exception: $f(R,G)$ with $f_{RR}f_{GG} - f_{RG}^2 = 0$, where $G = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$ is the Gauss-Bonnet invariant, does not possess ghosts but has other problems.

For $f_{RR}f_{GG} - f_{RG}^2 \neq 0$, a ghost was found very recently (A. De Felice and T. Tanaka, Progr. Theor. Phys. **124**, 503 (2010)).

Background FRW equations in $f(R)$ gravity

$$ds^2 = dt^2 - a^2(t)\left(dx^2 + dy^2 + dz^2\right)$$

$$H \equiv \frac{\dot{a}}{a}\,, \quad R = 6(\dot{H} + 2H^2)$$

The trace equation (4th order)

$$\frac{3}{a^3}\frac{d}{dt}\left(a^3\frac{df'(R)}{dt}\right) - Rf'(R) + 2f(R) = 8\pi G(\rho_m - 3p_m)$$

The 0-0 equation (3d order)

$$3H\frac{df'(R)}{dt} - 3(\dot{H} + H^2)f'(R) + \frac{f(R)}{2} = 8\pi G\rho_m$$

Conditions for viable $f(R)$ models

I. Conditions of classical and quantum stability:

$$f'(R) > 0 \ , \quad f''(R) > 0 \ .$$

Even the saturation of these inequalities should be avoided:
1. $f'(R_0) = 0$, $f''(R_0) \neq 0$: a generic anisotropic space-like curvature singularity forms.
Structure of the singularity in terms of a local Bianchi I type metric:

$$ds^2 = dt^2 - \sum_{i=1}^{3} a_i |t|^{2p_i} dx_i^2, \ \ u = s(2-s), \ \ s = \sum_i p_i, \ \ u = \sum_i p_i^2$$

with $R(0) = R_0$. No infinite number of BLH oscillations. The same structure but now with $R \propto |t|^{1-s} \to \infty$ for $1 < s < 2$ if $f(R) \propto R^2$ for $|R| \to \infty$. For a steeper behaviour of $f(R)$, the Big Rip occurs ($a \to \infty$, $s < 0$ for $|t| \to 0$).

2. $f''(R_0) = 0$, $f'''(R_0) \neq 0$: a weak singularity forms, loss of predictability of the Cauchy evolution.

$$a(t) = a_0 + a_1(t - t_s) + a_2(t - t_s)^2 + a_3|t - t_s|^{5/2} + \ldots\ .$$

The metric is C^2, but not C^3, continuous across this singularity, and there is no unambiguous relation between the coefficients a_3 for $t < t_s$ and $t > t_s$. Also, the equivalence of $f(R)$ gravity to scalar-tensor gravity with $\omega_{BD} = 0$ is broken in its vicinity.

However, weak removable singularities may exist, too. E.g. this occurs in the case $f''(R_0) = f'''(R_0) = 0$, $\quad f^{(4)}(R_0) \neq 0$.

II. Conditions for the existence of the Newtonian limit:

$$|F| \ll R, \quad |F'| \ll 1, \quad RF'' \ll 1$$

for $R \gg R_{now}$ and up to some very large R.
The same conditions for smallness of deviations from GR.
III. Laboratory and Solar system tests.
No deviation from the Newton law up to 50 μ.
No deviation from the Einstein values of the post-Newtonian coefficients β and γ up to 10^{-4} in the Solar system (and even ~ 4 times smaller for γ).
IV. Existence of a future stable (or at least metastable) de Sitter asymptote:

$$f'(R_{dS})/f''(R_{dS}) \geq R_{dS} \ .$$

Required since observed properties of DE are close to that of a cosmological constant.

V. Cosmological tests:
among them the anomalous growth of matter perturbations for recent redshifts

$$\left(\frac{\delta\rho}{\rho}\right)_m \propto t^{\frac{\sqrt{33}-1}{6}}$$

at the matter-dominated stage for $k \gg m_s(R)a$, where

$$m_s^2(R) = 1/3F''(R) \; .$$

Results in apparent discrepancy between the linear σ_8 and the primordial slope n_s estimated from CMB data (assuming GR) and from galaxy/cluster data separately.

VI. $f(R)$ cosmology should not destroy previous successes of present and early Universe cosmology in the scope of GR, including the existence of the matter-dominated stage driven by non-relativistic matter preceded by the radiation-dominated stage with the correct BBN and, finally, inflation.

Inflationary models in $f(R)$ gravity

1. The simplest one (Starobinsky, 1980):

$$f(R) = R + \frac{R^2}{6M^2}$$

with small one-loop quantum gravitational corrections producing the scalaron decay via the effect of particle-antiparticle creation by gravitational field (so all present matter is created in this way).
During inflation ($H \gg M$): $H = \frac{M^2}{6}(t_f - t), \quad |\dot{H}| \ll H^2$.

The only parameter M is fixed by observations – by the primordial amplitude of adiabatic (density) perturbations in the gravitationally clustered matter component:

$$M = 3.0 \times 10^{-6} M_{Pl} \; (50/N) \; ,$$

where $N \sim (50 - 55)$ is the number of e-folds between the first Hubble radius crossing during inflation of the present Hubble scale and the end of inflation, $M_{Pl} = \sqrt{G} \approx 10^{19}$ GeV.

Remains viable: $n_s = 1 - \frac{2}{N} \approx 0.96$, $r \equiv \frac{P_g}{P_\zeta} = \frac{12}{N^2} \approx 0.004$.

Observations: $n_s = 0.963 \pm 0.012$; $r < 0.17$ (95% CL).

The main and simplest alternative: the simplest scalar field inflationary model with $V(\phi) = \frac{m^2\phi^2}{2}$ and $m = M/\sqrt{2(1+r)} \approx 2.0 \times 10^{-6} M_{Pl}$ which produces the same n_s but the significantly larger $r = \frac{8}{N} \approx 0.15$.

2. Analogues of chaotic inflation: $F(R) \approx R^2 A(R)$ for $R \to \infty$ with $A(R)$ being a slowly varying function of R, namely

$$|A'(R)| \ll \frac{A(R)}{R}\,, \quad |A''(R)| \ll \frac{A(R)}{R^2}\,.$$

3. Analogues of new inflation, $R \approx R_1$:

$$F'(R_1) = \frac{2F(R_1)}{R_1}\,, \quad F''(R_1) \approx \frac{2F(R_1)}{R_1^2}\,.$$

Thus, all inflationary models in $f(R)$ gravity are close to the simplest one over some range of R.

One viable microphysical model leading to such form of $f(R)$

A non-minimally coupled scalar field with a large negative coupling ξ (for this choice of signs, $\xi_{conf} = \frac{1}{6}$):

$$L = \frac{R}{16\pi G} - \frac{\xi R\phi^2}{2} + \frac{1}{2}\phi_{,\mu}\phi^{,\mu} - V(\phi), \quad \xi < 0, \quad |\xi| \gg 1 \ .$$

Leads to $f' > 1$.
Recent development: the BEH (Higgs) inflation (F. Bezrukov and M. Shaposhnikov, 2008). In the limit $|\xi| \gg 1$, the BEH scalar tree level potential $V(\phi) = \frac{\lambda(\phi^2-\phi_0^2)^2}{4}$ just produces $f(R) = \frac{1}{16\pi G}\left(R + \frac{R^2}{6M^2}\right)$ with $M^2 = \lambda/24\pi\xi^2 G$ and $\phi^2 = |\xi|R/\lambda$ (for this model, $|\xi|G\phi_0^2 \ll 1$).

SM loop corrections to the tree potential leads to $\lambda = \lambda(\phi)$, then the same expression for $f(R)$ follows with

$$M^2 = \frac{\lambda(\phi(R))}{24\pi\xi^2 G}\left(1 + \mathcal{O}\left(\frac{d\ln\lambda(\phi(R))}{d\ln\phi}\right)^2\right).$$

.

The approximate shift invariance $\phi \to \phi + c,\ \ c = \mathrm{const}$ permitting slow-roll inflation for a minimally coupled inflaton scalar field transforms here to the approximate scale (dilatation) invariance
$\phi \to c\phi,\ \ R \to c^2 R,\ \ x^\mu \to x^\mu/c,\ \ \mu = 0, ..3$
in the physical (Jordan) frame. Of course, this symmetry needs not be fundamental, i.e. existing in some more microscopic model at the level of its action.

Simplifying the 3d order FRW equation for the $f(R) = R + (R^2/6M^2)$ vacuum model

I. Second order equation (Gurovich and Starobinsky, 1979).

$$f = (a\dot{a})^{3/2}, \; \xi = (12)^{-3/4} a^3$$

$$\frac{d^2 f}{d\xi^2} + \frac{M^2}{\xi^{2/3} f^{1/3}} = 0$$

Convenient for the analytical study of oscillations of R after the end of inflation.

II. First order equation (Starobinsky, 1980).

$$x = H^{3/2}, \ \ y = \frac{1}{2}H^{-1/2}\dot{H}, \ \ dt = \frac{dx}{3x^{2/3}y}$$

$$\frac{dy}{dx} = -\frac{M^2}{12x^{1/3}y} - 1$$

Convenient for drawing of the phase portrait. The y-axis corresponds to inflection points $\dot{a} = \ddot{a} = 0, \dddot{a} \neq 0$. A curve reaching the y-axis at the point $(0, y_0 < 0)$ continues from the point $(0, -y_0)$ to the right.

Late-time asymptotic:

$$a(t) \propto t^{2/3}\left(1 + \frac{2}{3Mt}\sin M(t - t_1)\right), \ \ R \approx -\frac{8M}{3t}\sin M(t-t_1)$$

$$< R^2 >= \frac{32M^2}{9t^2}, \ \ 8\pi G\rho_{s,eff} = \frac{3 < R^2 >}{8M^2} = \frac{4}{3t^2}$$

Valid in the presence of radiation and dust, too (with a smaller amplitude of oscillations in the latter case).

Present DE models in $f(R)$ gravity

Much more difficult to construct. The original proposal to make $f(R)$ diverging at $R \to 0$ does not work!

An example of the viable model satisfying the first 5 viability conditions (A. A. Starobinsky, JETP Lett. **86**, 157 (2007)):

$$f(R) = R + \lambda R_0 \left(\frac{1}{\left(1 + \frac{R^2}{R_0^2}\right)^n} - 1 \right)$$

with $n \geq 2$. $f(0) = 0$ is put by hand to avoid the appearance of a cosmological constant in the flat space-time.
Similar models:
1. W. Hu and I. Sawicki, Phys. Rev. D **76**, 064004 (2007).
2. A. Appleby and R. Battye, Phys. Lett. B **654**, 7 (2007).

No good microscopic justification for both the energy scale and the complicated form of $f(R)$ needed ($0 < f' < 1$).

Recent progress in $f(R)$ models of present DE

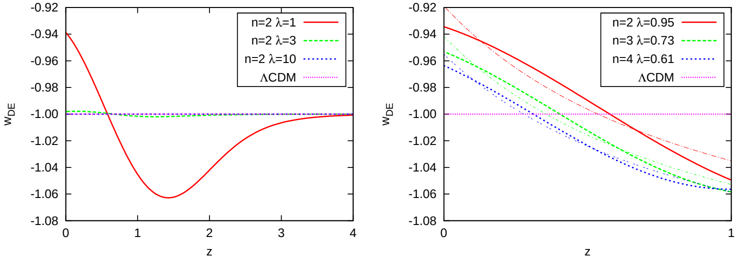
1. It was proved that viable models of DE typically exhibit phantom behaviour of dark energy during the matter-dominated stage and recent crossing of the phantom boundary $w_{\rm DE} = -1$. As a consequence of the anomalous growth of density perturbations in the cold dark matter + baryon component at recent redshifts, their growth index evolves non-monotonically with time and may even become negative temporarily (H. Motohashi, A. A. Starobinsky and J. Yokoyama, Progr. Theor. Phys. **123**, 887, 2010). Moreover, if the present mass of the scalaron is sufficiently large, there will be an infinite number of phantom boundary crossings during the future evolution of such cosmological models (H. Motohashi, A. A. Starobinsky and J. Yokoyama, JCAP **1106**, 006, 2011).

2. In such models of present DE, the sum of neutrino masses may be increased up to ~ 0.5 eV (H. Motohashi, A. A. Starobinsky and J. Yokoyama, Progr. Theor. Phys. **124**, 541 (2010)). Even more interesting, such models easily admit a 4th sterile neutrino with the mass ~ 1 eV (H. Motohashi, A. A. Starobinsky and J. Yokoyama, arXiv:1203.6828).

3. In order not to destroy any of previous successes of the early Universe cosmology, viable $f(R)$ models of present DE should be extended to large values of R with the $\sim R^2$ asymptotic behaviour and to negative R keeping $f'(R) > 0, \ f''(R) > 0$ at least up to the scale of inflation. Combined description of primordial and present DE using one $f(R)$ function is possible, but leads to completely different reheating after inflation during which strongly non-linear oscillations of R occur (S. A. Appleby, R. A. Battye and A. A. Starobinsky, JCAP **1006**, 005, 2010).

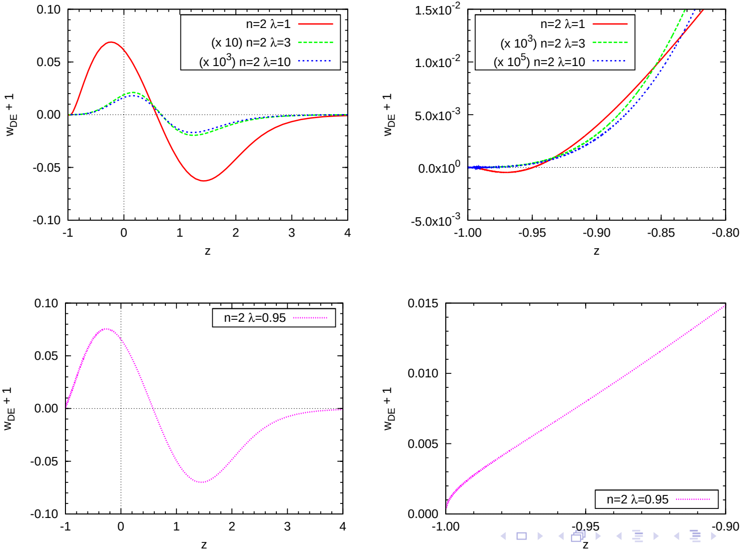
Phantom boundary crossing

Generic feature: phantom behaviour for $z > 1$,
crossing of the phantom boundary $w_{DE} = -1$ for $z < 1$.



Future evolution

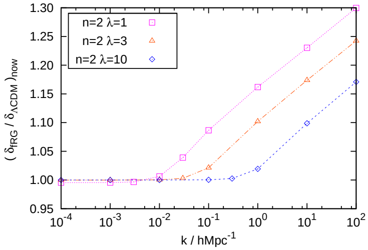
Infinite number of phantom boundary crossings at the stable future dS asymptote if $f'(R_{dS})/f''(R_{dS}) > 25R_{dS}/16$.

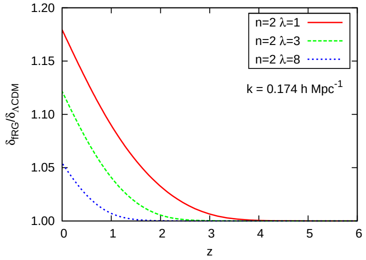


Anomalous growth of perturbations

Deeply in the sub-horizon regime:

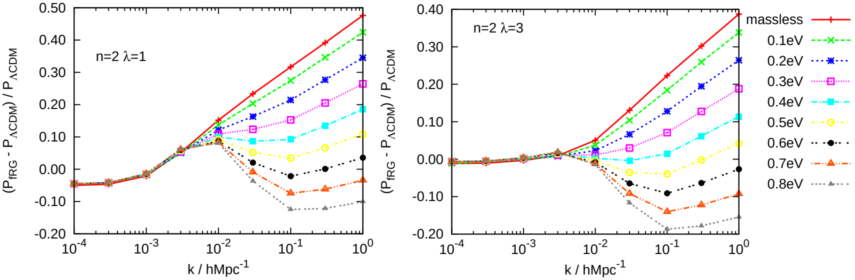
$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G_{eff}\,\rho\delta = 0\ , \quad G_{eff} = \frac{G}{f'}\,\frac{1 + 4\frac{k^2}{a^2}\frac{f''}{f'}}{1 + 3\frac{k^2}{a^2}\frac{f''}{f'}}\ .$$





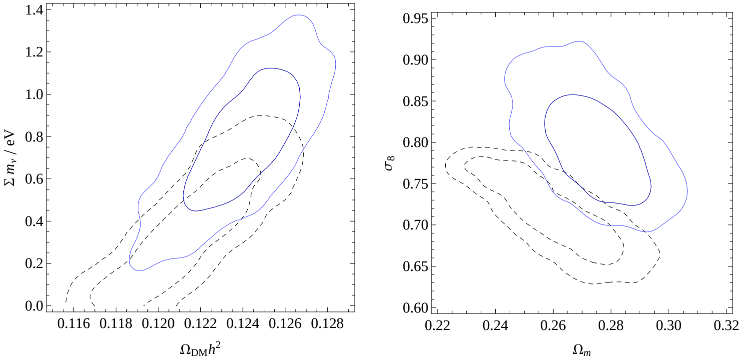
Massive standard neutrinos with $f(R)$ gravity

The anomalous growth of perturbations may be partially compensated by an increase of $\sum_\nu m_\nu$ as compared to the standard ΛCDM, up to $\mathcal{O}(0.5 \text{ eV})$.



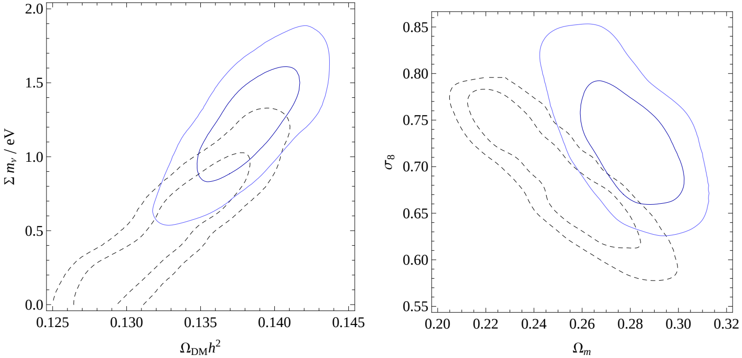
1 massive sterile neutrino with $f(R)$ gravity

Observational data from WMAP7 and SDSS DR7.



Best fit: $m_\nu = 0.860$ eV, $\chi^2_{eff} = 3767.0$ with $f(R)$ gravity versus $m_\nu = 0.109$ eV, $\chi^2_{eff} = 3774.1$ for the ΛCDM model.

2 massive sterile neutrinos with $f(R)$ gravity



About 5% less σ_8 compared to the 1 sterile neutrino case.

Structure of corrections to GR

$$R = R^{(0)} + \delta R_{ind} + \delta R_{osc} \ ,$$

$$R^{(0)} = 8\pi G T_m \propto a^{-3} \ ,$$

$$\delta R_{ind} = \left(RF'(R) - 2F(R) - 3\nabla_\mu \nabla^\mu F'(R)\right)_{R=R^{(0)}} \ ,$$

$$R \gg R_0, \quad \delta R_{ind} \approx \mathrm{const} = -F(\infty) = 4\Lambda(\infty) \ .$$

No Dolgov-Kawasaki instability.

$$MD: \qquad \delta R_{osc} \propto t^{-(3n+4)} \sin\left(c_1 t^{-(2n+1)} + c_2\right),$$

$$RD: \quad \delta R_{osc} \propto t^{-3(3n+4)/4} \sin\left(c_3 t^{-(3n+1)/2} + c_4\right).$$

$\delta a/a$ is small but $\delta R_{osc}/R^{(0)}$ diverges for $t \to 0$.

δR_{osc} should be very small just from the beginning – a problem for those $f(R)$ models which do not let R become negative due to crossing of the $f''(R) = 0$ point.

The "scalaron overproduction" problem.

Three new problems

In the early Universe:

- Unlimited growth of $m_s(R)$ for $t \to 0$: when $m_s(R)$ exceeds M_{Pl}, quantum-gravitational loop corrections invalidate the use of an effective quasi-classical $f(R)$ gravity.
- Unlimited growth of the amplitude of δR oscillations for $t \to 0$ (the "scalaron overproduction" problem).
- "Big Boost" singularity before the Big Bang:

$$a(t) = a_0 + a_1(t - t_0) + a_2|t - t_0|^k + ..., \ 1 < k = \frac{4n+1}{2n+1} < 2\,,$$

if $|F(R) - F(\infty)| \propto R^{-2n}$ for $R \to \infty$, so $f''(\infty) = 0$.

Curing all three problems

S. A. Appleby, R. A. Battye and A. A. Starobinsky,
JCAP **1006**, 005 (2010).

Add $\frac{R^2}{6M^2}$ to $f(R)$ with M not less than the scale of inflation. Then the first and and third problems go away. The second problem still remains, but (any) inflation can solve it. However, in all known inflationary models R may be negative during reheating after inflation (e.g. when $V(\phi) = 0$). Necessity of an extension of $f(R)$ to $R < 0$ keeping $f''(R) > 0$. As a result, a non-zero g-factor $(0 < g < 1/2)$ arises:

$$g = \frac{f'(R) - f'(-R)}{2f'(R)} , \quad R_0 \ll R \ll M^2 .$$

An example satisfying all 6 viability conditions: the g-extended R^2-corrected AB model

$$f(R) = (1-g)R + g\epsilon \log\left[\frac{\cosh\left(R/\epsilon - b\right)}{\cosh b}\right] + \frac{R^2}{6M^2} \ .$$

$m_s \approx M = const$ for $\rho_m \gg 10^{-27}$ g/cm^3 –
no "chameleon" behaviour in laboratory and Solar system experiments.

The same can be done for HSS-type models (H. Motohashi, A. A. Starobinsky and J. Yokoyama, in preparation).

Combined models of primordial and present DE

Construction of a viable model of present dark energy in $f(R)$ gravity naturally leads to combined models of primordial and present DE.

However, to take $f(R)$ simply as some function for which the equation $Rf'(R) = 2f(R)$ has 2 roots is greatly insufficient! What should be achieved in addition:
1) metastability of inflation;
2) sufficiently fast decay of the scalaron into matter quanta after inflation (thus, these models are *not* classical vacuum $f(R)$ models);
3) validity of the stability conditions $f' > 0, \;\; f'' > 0$ during all stages from inflation up to the present time.

If $M \approx 3 \times 10^{-6} M_{Pl}$, the scalaron can play the role of an inflaton, too. Then the inflationary predictions are formally the same as for the pure $R + R^2/6M^2$ inflationary model which does not describe the present DE:

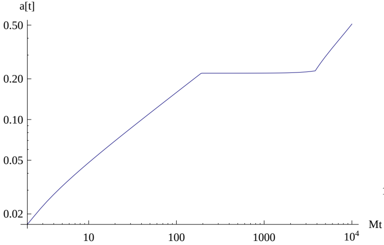
$$n_s = 1 - \frac{2}{N}\ , \quad r = \frac{12}{N^2}.$$

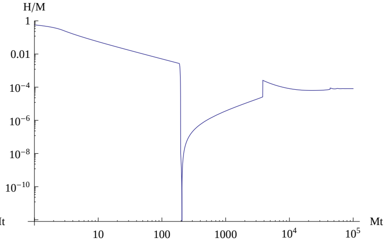
However, N is different, $N \sim 70$ for the unified model (versus $N \sim (50-55)$ for the purely inflationary one) because the stage if reheating after inflation becomes completely different: it consists of unequal periods with $a \approx const$ and $a \propto t^{1/2}$.
Duration of the periods in terms of $\ln t$:
$-\ln(1-2g)$ and $-2\ln(1-2g)$ respectively.
So, $a(t) \propto t^{1/3}$ on average for a long time after the end of inflation, in contrast to

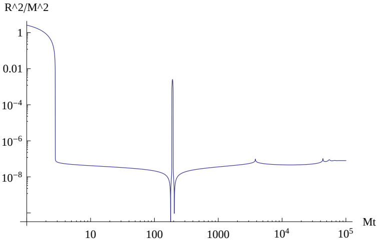
$$a(t) \propto t^{2/3}\left(1 + \frac{2}{3Mt}\sin M(t-t_1)\right)$$

for the pure inflationary $f(R) = R + R^2/6M^2$ model.

Observable prediction which is, however, degenerate with other inflationary models in $f(R)$ gravity.

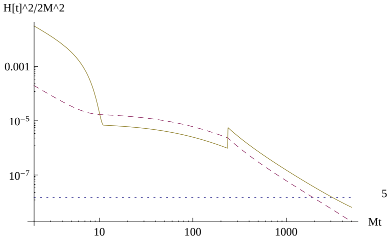

a[t]
0.50
0.20
0.10
0.05
0.02
10
100
1000
10^4
Mt

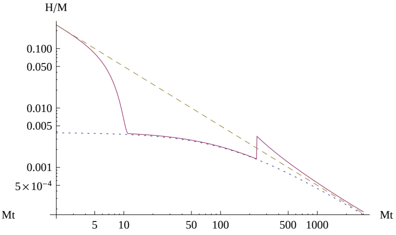

H/M
1
0.01
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10
100
1000
10^4
10^5
Mt


R^2/M^2
1
0.01
10^{-4}
10^{-6}
10^{-8}
10
100
1000
10^4
10^5
Mt

Reheating – due to gravitational particle creation which occurs mainly at the end of inflation. Less efficient than in the pure inflationary $f(R) = R + R^2/6M^2$ model,

$$t = t_{\rm reh} \sim M^{-4} M_{\rm Pl}^3 \sim 10^{-18}\ {\rm s}\ .$$





Conclusions

- In the range of R where $f'(R) > 0, \quad f''(R) > 0$, $f(R)$ gravity is a good macroscopic theory of gravity without tachyons, ghosts and weak singularities preventing the deterministic Cauchy evolution.
- No new stationary asymptotically-flat black hole solutions different from the Kerr-de Sitter one. However, in principle relaxation to this solution may be longer than in GR with a cosmological term.
- More interesting applications in cosmology. The simplest non-trivial case $f(R) = R + R^2/6M^2$ already produces a metastable slow-roll inflationary stage. Its post-inflationary evolution averaged over time intervals $\Delta t \gg M^{-1}$ is dust-like.

- Supplementing this classical $f(R)$ model by small one-loop quantum gravitational corrections which produce a) generation of scalar and tensor perturbations during inflation and b) the scalaron decay after inflation via the effect of particle-antiparticle creation by gravitational field, leads to a viable inflationary model (a simplified variant of the pioneer 1980 inflationary model) predicting the slope of the spectrum of primordial density perturbations $n_s = 1 - \frac{2}{N} \approx 0.96$ in agreement with observations. This model describes the gravitational sector of the Higgs inflation, too.
- Its critical test: the low value for the tensor-to-scalar ratio of primordial metric perturbations $r = 12/N^2 \approx 0.4\%$.

- Other possible viable inflationary models in $f(R)$ gravity are functionally close to this model over the range of R for which inflation occurs. To have a graceful exit to the radiation-dominated FRW stage after scalaron decay and heating of matter, the function $f(R)$ should be analytic and satisfy the stability conditions $f'(R) > 0, \ f''(R) > 0$ for all $|R|$ less than the scale of inflation including $R = 0$.
- Much more problems with models of present DE. Still a narrow class of $f(R)$ models of present DE remains viable: it is possible to construct predictive models satisfying all existing cosmological, Solar system and laboratory data, and distinguishable from ΛCDM. However, these models require a complicated structure of $f(R)$ at low R for which no microscopic explanation is known at present.

- The most critical test for all $f(R)$ models of present dark energy: anomalous growth of density perturbations in the matter component at recent redshifts $z \sim 1-3$. A number of different ways to check it in the linear and non-linear regimes.
- In these models, a 4th sterile neutrino with the mass ~ 1 eV is permitted and produces even a slightly better fit to existing observational data as compared to the standard ΛCDM model, without lowering the best fit value for σ_8.
- In order not to destroy all previous successes of the early Universe cosmology, these viable $f(R)$ models of present DE should be extended to large R with the $\sim R^2$ asymptotic behaviour and to negative R keeping $f'(R) > 0, \; f''(R) > 0$ at least up to the scale of inflation. This results in a constant scalaron mass for laboratory and higher values of R – natural range for choosing a normalization point for one-loop quantum corrections.

- This naturally (though not inevitably) leads to combined models of primordial and present DE for the specific choice of M: $M \approx 3 \times 10^{-6} M_{Pl}$. However, combined inflationary – DE $f(R)$ models have a significantly different behaviour after inflation as compared to pure inflationary $f(R)$ models, with strongly non-linear oscillations of the scale factor $a(t)$. The ultimate reason for this: different values of G_{eff} for $R > 0$ and $R < 0$ due to $f''(R) > 0$.