Singular General Relativity

How I learned to stop worrying and love the singularities

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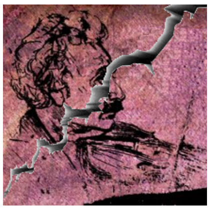
1 Introduction: Do the singularities predict the breakdown of General Relativity?



It is often said that General Relativity predicts its own breakdown, by predicting the occurrence of singularities. In addition, when trying to quantize gravity, it appears to be perturbatively nonrenormalizable.

• Are these problems signs that we should give up General Relativity in exchange for more radical approaches (superstrings, loop quantum gravity *etc.*)?

• Are these limits of GR, or of our tools?

• What if understanding the singularities also helps with the problem of quantization?[1]

2 The cure for singularities

2.1 Singular semi-Riemannian geometry

The methods of *singular semi-Riemannian geometry* (see [2, 3] and the appendix) allow us to:

• find mathematical descriptions of singularities, and understand them • replace the singular quantities with regular ones, which are equivalent to them if $\det g \neq 0$ • write non-singular field equations, for example extensions of Einstein's equations

2.2 Singular General Relativity

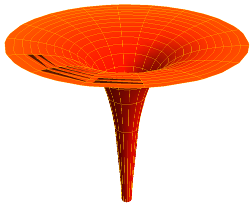
Singular General Relativity is the application of the methods of singular semi-Riemannian geometry to General Relativity. For this, the metric is required to be *benign*, but often a metric which is apparently *malign* can be put in a form which is benign, by a coordinate change.

• *Malign singularity*: $g_{ab} \to \infty$. *Benign singularity*: g is smooth and $\det g = 0$.

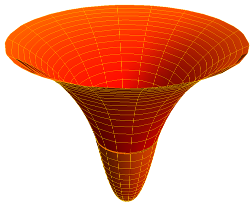
[1]Usually is considered that when GR will be quantized, this will solve the singularities too, by showing probably that quantum fields prevent the occurrence of singularities. Here we will explore the opposite view.

3 Results: Smoothening black hole singularities

Apparently, the black hole singularities are malign, because some of the components of the metric are divergent ($g_{ab} \to \infty$). For the standard stationary black holes, we can find transformations of coordinates which make them smooth. This also makes, for charged black holes, the electromagnetic potential and field smooth. The metric is put in a form which allows the evolution equations to go beyond the singularities.



Malign singularities $g_{ab} \to \infty$



Benign singularities g_{ab} smooth, $\det g \to 0$

3.1 Schwarzschild black hole

In Schwarzschild coordinates

$$\mathrm{d}s^2 = -\frac{r-2m}{r}\mathrm{d}t^2 + \frac{r}{r-2m}\mathrm{d}r^2 + r^2\mathrm{d}\sigma^2,$$

where $\mathrm{d}\sigma^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2$.

In non-singular coordinates [5]

$$\begin{aligned}\mathrm{d}s^2 &= -\frac{4\tau^4}{2m-\tau^2}\mathrm{d}\tau^2 \\ &\quad +(2m-\tau^2)\tau^{2T-4}\left(T\xi\mathrm{d}\tau + \tau\mathrm{d}\xi\right)^2 + \tau^4\mathrm{d}\sigma^2,\end{aligned}$$

where $(r,t) \mapsto (\tau^2, \xi\tau^T)$, $T \geq 2$.

3.2 Reissner-Nordström black hole

In Reissner-Nordström coordinates

$$\mathrm{d}s^2 = -\frac{\Delta}{r^2}\mathrm{d}t^2 + \frac{r^2}{\Delta}\mathrm{d}r^2 + r^2\mathrm{d}\sigma^2$$

where $\Delta = r^2 - 2mr + q^2$.

The electromagnetic potential is singular:

$$A = -\frac{q}{r}\mathrm{d}t$$

In non-singular coordinates [6]

$$\begin{aligned}\mathrm{d}s^2 &= -\Delta\rho^{2T-2S-2}\left(\rho\mathrm{d}\tau + T\tau\mathrm{d}\rho\right)^2 \\ &\quad +\frac{S^2}{\Delta}\rho^{4S-2}\mathrm{d}\rho^2 + \rho^{2S}\mathrm{d}\sigma^2,\end{aligned}$$

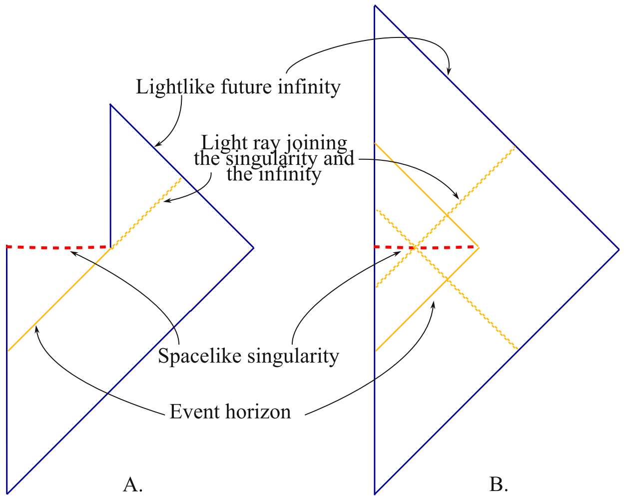
where $(t,r) \mapsto (\tau\rho^T, \rho^S), T > S \geq 1$.

The electromagnetic potential is non-singular:

$$A = -q\rho^{T-S-1}\left(\rho\mathrm{d}\tau + T\tau\mathrm{d}\rho\right)$$

4 Information paradox

Penrose-Carter diagrams for non-rotating and electrically neutral evaporating black holes.



A. If the singularity is malign, entering *information is lost*. When accounting for Quantum Mechanics, this causes in particular a *violation of unitarity*, if a system which is entangled with another system is lost in the singularity. **B.** If the singularity is benign, the field equations can be extended beyond the singularity, and the information is preserved [5–8].

5 Einstein equation at singularities

Singular semi-Riemannian geometry (see appendix and [2,3]) introduces generalizations of semi-Riemannian spacetimes which admit degenerate metric but have smooth Riemann curvature R_{abcd}. But the Einstein tensor is usually singular. Yet, in 4D we can rewrite the Einstein equation in terms of quantities which remain smooth at singularities. The resulting equations are equivalent to Einstein's so long as the metric is regular, but work as well at singularities.

5.1 Densitized Einstein equation

On 4D semi-regular spacetimes the Riemann curvature R_{abcd} is smooth. Hence the Einstein tensor density $G \det g$ is smooth too, being:

$$G_{ab} \det g = g_{kl} \epsilon_a{}^{kst} \epsilon_n{}^{lpq} R_{stpq}.$$

We can write a densitized version of the Einstein equation:

$$G_{ab} \det g + \Lambda g_{ab} \det g = \kappa T_{ab} \det g,$$

For regular metric it is equivalent to Einstein's equation, but works at singularities too [2].

5.2 Expanded Einstein equation

A quasi-regular spacetime is a 4D semi-regular manifold with smooth Ricci decomposition

$$R_{abcd} = \frac{1}{12} R (g \circ g)_{abcd} + \frac{1}{2} (S \circ g)_{abcd} + C_{abcd}$$

where $S_{ab} := R_{ab} - \frac{1}{4} R g_{ab}$, and

$$(h \circ k)_{abcd} := h_{ac} k_{bd} - h_{ad} k_{bc} + h_{bd} k_{ac} - h_{bc} k_{ad}.$$

In this case we can write the *expanded Einstein equation* [11]:

$$(G \circ g)_{abcd} + \Lambda (g \circ g)_{abcd} = \kappa (T \circ g)_{abcd}.$$

The FLRW spacetime is the warped product $I \times_a \Sigma$, $a : I \to \mathbb{R}$, $a \geq 0$, with metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t) \mathrm{d}\Sigma^2$$

where usually Σ is $S^3, \mathbb{R}^3$, or H^3 [3].
The energy density ρ and pressure density p are singular at the Big-Bang, where $a(t) = 0$:

$$\rho = \frac{3}{\kappa} \frac{\dot{a}^2 + k}{a^2}, \quad \rho + 3p = -\frac{6}{\kappa} \frac{\ddot{a}}{a}.$$

But in terms of the correct densities

$$\widetilde{\rho} = \rho \sqrt{-g}, \quad \widetilde{p} = p \sqrt{-g},$$

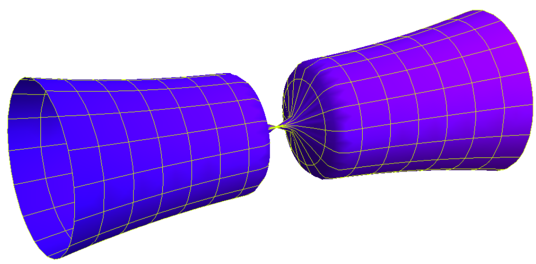
the equations become smooth:

$$\widetilde{\rho} = \frac{3}{\kappa} a \left(\dot{a}^2 + k \right) \sqrt{g_\Sigma}, \quad 3\widetilde{p} + \widetilde{\rho} = -\frac{6}{\kappa} a^2 \ddot{a} \sqrt{g_\Sigma}.$$

Hence, $\widetilde{\rho}$ and $\widetilde{p}$ are smooth, as it is the densitized stress-energy tensor

$$T_{ab} \sqrt{-g} = (\widetilde{\rho} + \widetilde{p}) \, u_a u_b + \widetilde{p} g_{ab}.$$

These singularities are quasi-regular [9, 10].



6 Weyl curvature hypothesis

The *Weyl curvature hypothesis* was proposed by R. Penrose, to account for the high homogeneity and low entropy characterizing the Big-Bang. It states that at the Big-Bang singularity, the Weyl curvature tensor $C_{abcd} = 0$.

At a quasi-regular Big-Bang singularity, C_{abcd} is smooth. At the singularity it vanishes, because it lives in $T^\bullet M$, which has dimension ≤ 3 at singularities [12].

7 Dimensional reduction and Quantum Gravity

Different results suggest that a dimension < 4 may act like a dimensional regulator for QFT and for QG. In lower dimension, the Weyl tensor vanishes, and the local degrees of freedom (gravitational waves and gravitons) disappear, allowing QG to be renormalizable.

Benign singularities undergo a dimensional reduction, and may be the needed dimensional regularizers for QG:

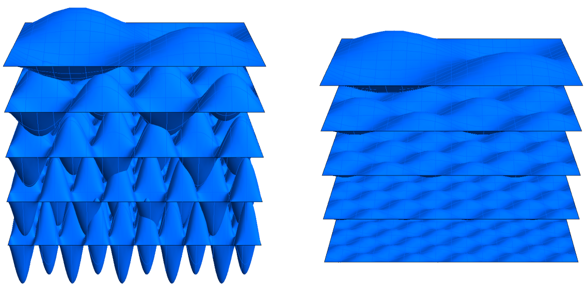
- g_{ab} is independent on some directions.
- $\sqrt{-\det g} \to 0$, reducing the contribution of lower distance Feynman integrals.
- The admissible fields live in lower dimension spaces:

$$\operatorname{rank} g_p = \dim T_{p\bullet}M = \dim T_p{}^\bullet M < 4.$$

- For quasi-regular singularities $C_{abcd} = 0$.

The quantities $\det g$ and C_{abcd} vanish as approaching singularities. But QG needs them to decrease with the scale. They may do this:

As the energy approaches the UV limit, the number of the particles in Feynman diagrams increases. If particles are benign singularities, this means lower $\det g$ and C_{abcd}. We conjecture: this gives the needed regularization [13].



A Singular semi-Riemannian Geometry

- A *singular semi-Riemannian manifold* (M, g) is a differentiable manifold M with a symmetric bilinear form g, named *metric*, on the tangent bundle TM. The metric g may be indefinite, and may be degenerate (*i.e.* $\det g = 0$ at some points). It may have constant or variable signature.
- *Malign singularity*: $g_{ab} \to \infty$. *Benign singularity*: g is smooth and $\det g = 0$.

A.1 The problem

The geometric quantities can't be defined:

$$\Gamma^c{}_{ab} = \frac{1}{2} g^{cs}(\partial_a g_{bs} + \partial_b g_{sa} - \partial_s g_{ab})$$

$$R^d{}_{abc} = \Gamma^d{}_{ac,b} - \Gamma^d{}_{ab,c} + \Gamma^d{}_{bs}\Gamma^s{}_{ac} - \Gamma^d{}_{cs}\Gamma^s{}_{ab}$$

$$R_{ab} = R^s{}_{asb}, \; R = g^{pq}R_{pq}$$

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}$$

These quantities are not usually defined even if g_{ab} are finite, since $g^{ab} \to \infty$ when $\det g \to 0$.

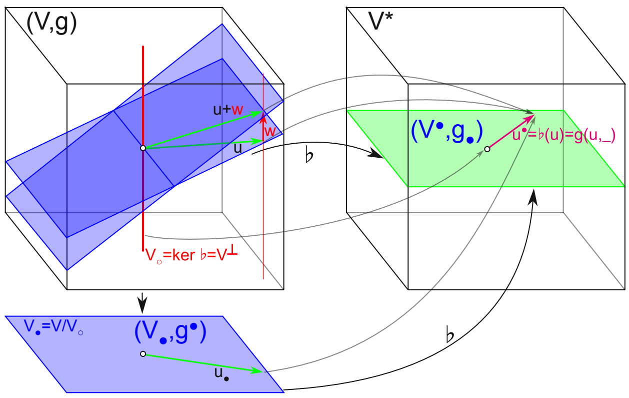
A.2 The idea of a solution

Use non-singular quantities, equivalent to the singular ones for non-degenerate metric g.

Singular	Non-Singular	When g is...
$\Gamma^c{}_{ab}$ (2^{nd})	Γ_{abc} (1^{st})	smooth
$R^d{}_{abc}$	R_{abcd}	semi-regular
R_{ab}	$R_{ab}\sqrt{\lvert\det g\rvert}^W$, $W \leq 2$	semi-regular
R	$R\sqrt{\lvert\det g\rvert}^W$, $W \leq 2$	semi-regular
R_{ab}	$\operatorname{Ric} \circ g$	quasi-regular
R	$Rg \circ g$	quasi-regular

But how to define them?

A.3 Degenerate metric and covariant contraction

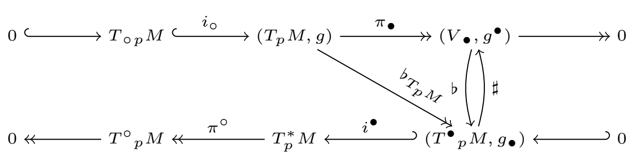


(V, g) is an inner product vector space. The morphism $\flat : V \to V^*$ is defined by $u \mapsto u^\bullet := \flat(u) = u^\flat = g(u, \_)$. The radical $V_\circ := \ker \flat = V^\perp$ is the set of isotropic vectors in V. $V^\bullet := \operatorname{im} \flat \leq V^*$ is the image of $\flat$. The inner product g induces on $V^\bullet$ an inner product defined by $g_\bullet(u_1^\flat, u_1^\flat) := g(u_1, u_2)$, which is the inverse of g iff $\det g \neq 0$. The quotient $V_\bullet := V/V_\circ$ consists in the equivalence classes of the form $u + V_\circ$. On $V_\bullet$, g induces an inner product $g^\bullet(u_1 + V_\circ, u_2 + V_\circ) := g(u_1, u_2)$.

- Once we have defined the reciprocal inner product $g_\bullet(\omega, \tau)$, we can define covariant contraction for 1-forms from $V^\bullet$. Then we define it for tensors with two covariant indices living in $V^\bullet$.

$$T(\omega_1, \ldots, \bullet, \ldots, \bullet, \ldots, \omega_k)$$

- We extend these definitions to a singular semi-Riemannian manifold (M, g).



A.4 Covariant derivative

• The covariant derivative $\nabla_X Y$ can't be defined. We can use instead the *Koszul form*:

$$\begin{aligned}\mathcal{K}(X,Y,Z) \;:=\; & \frac{1}{2}\{X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle \\ & -\langle X,[Y,Z]\rangle + \langle Y,[Z,X]\rangle + \langle Z,[X,Y]\rangle\}\end{aligned}$$

• For non-degenerate metric $\nabla_X Y = \mathcal{K}(X,Y,\_)^\sharp$.

• The Christoffel's symbols of the first kind are $\Gamma_{abc} = \mathcal{K}(\partial_a,\partial_b,\partial_c) = \frac{1}{2}(\partial_a g_{bc} + \partial_b g_{ca} - \partial_c g_{ab})$

• The *lower covariant derivative*: $(\nabla^\flat_X Y)(Z) := \mathcal{K}(X,Y,Z)$, for any $Z \in \mathfrak{X}(M)$.

A singular semi-Riemannian manifold is *radical-stationary* if $\mathcal{K}(X,Y,\_) \in \mathcal{A}^\bullet(M) := \Gamma(T^\bullet M)$.

• *Covariant derivative of covariant indices:*

$(\nabla_X \omega)(Y) := X(\omega(Y)) - g_\bullet(\nabla^\flat_X Y, \omega)$

$$\begin{aligned}(\nabla_X T)\quad & (Y_1,\dots,Y_k) = X(T(Y_1,\dots,Y_k)) \\ & -\textstyle\sum_{i=1}^k \mathcal{K}(X,Y_i,\bullet)T(Y_1,\dots,\bullet,\dots,Y_k)\end{aligned}$$

• A singular *semi-regular* manifold is a radical-stationary manifold so that $\nabla_X \nabla^\flat_Y Z \in \mathcal{A}^\bullet(M)$.

• *Isotropic singularities.* A manifold $(M,\Omega^2\tilde{g})$, where $\tilde{g}$ is a non-degenerate metric, and Ω a smooth function, is semi-regular.

A.5 Curvature

Let (M,g) be radical-stationary.

• The *lower Riemann curvature operator* is: $\mathcal{R}^\flat_{XY}Z := \nabla_X\nabla^\flat_Y Z - \nabla_Y\nabla^\flat_X Z - \nabla^\flat_{[X,Y]}Z$

• The *Riemann curvature tensor* is: $R(X,Y,Z,T) := (\mathcal{R}^\flat_{XY}Z)(T)$

$R_{abcd} = \partial_a\Gamma_{bcd} - \partial_b\Gamma_{acd} + (\Gamma_{ac\bullet}\Gamma_{bd\bullet} - \Gamma_{bc\bullet}\Gamma_{ad\bullet})$

• The Ricci tensor: $\mathrm{Ric}(X,Y) := R(X,\bullet,Y,\bullet)$

• The scalar curvature $s := \mathrm{Ric}(\bullet,\bullet)$.

• If (M,g) is semi-regular, the Riemann curvature is a smooth tensor field, but the Ricci and scalar curvatures may be singular.

A.6 Singular warped products

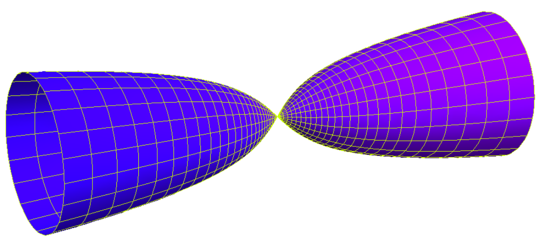
The *warped product* of two singular semi-Riemannian manifolds (B,g_B) and (F,g_F), with *warping function* $f \in \mathscr{F}(B)$ is the singular semi-Riemannian manifold

$$B \times_f F := \big(B \times F, \pi^*_B(g_B) + (f\circ\pi_B)\pi^*_F(g_F)\big),$$

where $\pi_B : B\times F \to B$ and $\pi_F : B\times F \to F$ are the canonical projections.

The inner product on $B\times_f F$ takes the form

$$\mathrm{d}s^2 = \mathrm{d}s^2_B + f^2\mathrm{d}s^2_F.$$



The usual definition of warped product requires (B,g_B) and (F,g_F) to be non-degenerate, and $f>0$. Here we allow it to be $f\geq 0$, including by this possible singularities.

• If (B,g_B) and (F,g_F) are radical-stationary and $\mathrm{d}f \in \mathcal{A}^\bullet(B)$, the warped product manifold $B\times_f F$ is radical-stationary.

• If (B,g_B) and (F,g_F) are semi-regular, and $\mathrm{d}f \in \mathscr{A}^{\bullet 1}(B)$, the warped product manifold $B\times_f F$ is semi-regular.

• Example of warped product: the Friedmann-Lemaître-Robertson-Walker spacetime.

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